A class of warped filter bank frames tailored to non-linear frequency scales

Nicki Holighaus[a,*], Christoph Wiesmeyr[b], Zdeněk Průša[a]

[a]*Acoustics Research Institute Austrian Academy of Sciences, Wohllebengasse 12–14, A-1040 Vienna, Austria*
[b]*AIT Austrian Institute of Technology GmbH, Donau-City-Strasse 1, A-1220 Vienna, Austria and NuHAG, Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria*

Abstract

A method for constructing non-uniform filter banks is presented. Starting from a uniform system of translates, generated by a prototype filter, a non-uniform covering of the frequency axis is obtained by composition with a warping function. The warping function is a $\mathcal{C}^1$-diffeomorphism that determines the frequency progression and can be chosen freely, apart from minor technical restrictions. The resulting functions are interpreted as filter frequency responses and, combined with appropriately chosen decimation factors, give rise to a non-uniform analysis filter bank. Classical Gabor and wavelet filter banks are obtained as special cases. Beyond the state-of-the-art, we construct a filter bank adapted to a frequency scale derived from human auditory perception and families of filter banks that can be interpreted as an interpolation between linear (Gabor) and logarithmic (wavelet) frequency scales. For any arbitrary warping function, we derive straightforward decay conditions on the prototype filter and bounds for the decimation factors, such that the resulting warped filter bank forms a frame. In particular, we obtain a simple and constructive method for obtaining tight frames with bandlimited filters by invoking previous results on generalized shift-invariant systems.

Keywords: time-frequency; adaptive systems; frames; generalized shift-invariant systems; non-uniform filter banks; warping

1. Introduction

In this contribution, we introduce a class of non-uniform time-frequency systems optimally adapted to non-linear frequency scales. The central paradigm of our construction, and what distinguishes it from previous approaches, is to provide uniform frequency resolution *on the target frequency scale*. Invertible time-frequency systems are of particular importance, since they allow for stable recovery of signals from the time-frequency representation coefficients. Therefore, we also derive necessary and sufficient conditions for the resulting systems to form a frame.

To demonstrate the flexibility and importance of our construction, illustrative examples recreating (or imitating) classical time-frequency representations such as Gabor [32, 34, 29, 30], wavelet [44, 15] or α-transforms [12, 31, 47] are provided. While this paper considers the setting of (discrete)

*Corresponding author

Email addresses: nicki.holighaus@oeaw.ac.at (Nicki Holighaus), christoph.wiesmeyr@ait.ac.at (Christoph Wiesmeyr), zdenek.prusa@oeaw.ac.at (Zdeněk Průša)

Hilbert space frames, the properties of continuous warped time-frequency systems are investigated in the related contribution [40]. Whenever a time-frequency filter bank adapted to a given frequency progression and with linear time-progression in each channel is desired, we believe that the proposed *warped filter banks* provide the right framework for its design.

In the proposed method, generalized shift-invariant (GSI) systems [56, 37, 10, 3] over $\mathbf{L}^2(\mathbb{R})$ are constructed from a prototype frequency response θ via composition with a *warping function* that specifies the desired frequency scale/progression. To highlight the relation of the resulting warped time-frequency systems to non-uniform filter banks, we use terminology from filter bank theory and refer to GSI systems as (analysis) filter banks, despite operating on $\mathbf{L}^2(\mathbb{R})$ instead of the sequence space $\ell^2(\mathbb{Z})$.

It will be shown that warped time-frequency systems provide a very natural and intuitive framework for time-frequency analysis on non-linear frequency scales. Most importantly, invertible systems are constructed with ease, in particular tight filter bank frames with bandlimited filters can be obtained through a very simple procedure. Moreover, the selection of appropriate decimation factors (sampling steps) is simplified by the filter bandwidths' direct link to the derivative of the warping function, leading to a canonical choice of *natural decimation factors*. These decimation factors are determined for all frequency channels simultaneously by the selection of a single parameter $\tilde{a} > 0$. Importantly, we show that natural decimation factors and expected decay conditions on the prototype θ ensure that a warped filter bank is a Bessel sequence and satisfies the important *local integrability condition* for GSI systems, see [37, 10]. Additionally, under these conditions, there is a safe region of small enough choices for $\tilde{a}$, such that the resulting warped filter bank is guaranteed to be a frame. The later sections of this contribution are concerned with providing some examples for the given abstract framework, as well as its adaptation to digital signals in $\ell^2(\mathbb{Z})$, see also [39], where warped filter banks for $\ell^2(\mathbb{Z})$ were first presented.

Adapted time-frequency systems. Time-frequency (or time-scale) representations are an indispensable tool for signal analysis and processing. The most widely used and most thoroughly explored such representations are certainly Gabor and wavelet transforms and their variations, e.g. windowed modified cosine [53, 54] or wavelet packet [11, 64] transforms. The aforementioned transforms unite two very important properties: There are various, well-known necessary and/or sufficient conditions for stable inversion from the transform coefficients, i.e. for the generating function system to form a frame. In addition to the perfect reconstruction property, the frame property ensures stability of the synthesis operation after coefficient modification, enabling controlled time-frequency processing. Furthermore, efficient algorithms for the computation of the transform coefficients and the synthesis operation exist for each of the mentioned transforms [60, 44].

While providing a sound and well-understood mathematical foundation, Gabor and wavelet transforms are designed to follow two specified frequency scales: linear, respectively logarithmic. A wealth of approaches exists to soften this restriction, e.g. decompositions using filter banks [13, 14, 62, 6], for example based on perceptive frequency scales [36, 59, 52]. Adaptation over time is considered in approaches such as modulated lapped transforms [45], adapted local trigonometric transforms [63] or (time-varying) wavelet packets [55]. Techniques that jointly offer flexible time-frequency resolution and variable redundancy, the perfect reconstruction property and efficient computation are scarce however. The setting of so-called nonstationary Gabor transforms [3], a recent generalization of classical Gabor transforms, provides the latter 2 properties while allowing for freely chosen time progression and varying resolution. In this construction, the frequency scale is still linear, but the sampling density may be changed over time. The properties of nonstationary Gabor systems are a matter of ongoing investigation, but a number of results already exist [38, 19, 18].

When desiring increased flexibility along frequency, generalized shift-invariant systems [56, 37, 8, 7, 10], or equivalently (non-uniform) *filter banks* [2], provide the analogous concept. They offer full flexibility in frequency, with a linear time progression in each filter, but flexible sampling density across the filters. Analogous, continuously indexed systems are considered in [58, 41]. Indeed, nonstationary Gabor systems are equivalent to filter banks via an application of the (inverse) Fourier transform to the generating functions. Note that all the widely used transforms mentioned in the previous paragraph can be interpreted as filter banks.

Adaptation to non-linear frequency scales through warping. There have been previous attempts to construct adapted filter banks by frequency warping. All previous methods have in common, however, that they focus on unitary warping operators that cannot provide the shape-preserving property that is central to our approach. Therefore, the properties of the resulting systems and the challenges faced in their construction are quite different.

For example, Braccini and Oppenheim [51], as well as Twaroch and Hlawatsch [61], propose a unitary warping of a collection system of translates, interpreted as filter frequency responses. In [51] only spectral analysis is desired, while time-frequency distributions are constructed in [61], without considering signal reconstruction.

The application of unitary warping to an entire Gabor or wavelet system has also been investigated [5, 4, 23, 24]. Although unitary transformation bequeaths basis (or frame) properties to the warped atoms, the resulting system is not anymore a filter bank. Instead, the warped system produces undesirable, dispersive time-shifts and the resulting representation is not easily interpreted, see [23]. Only for the continuous short-time Fourier transform, or under quite strict assumptions on a Gabor system, a *redressing* procedure can be applied to recover a GSI system [21]. In all other cases, the combination of unitary warping with redressing complicates the efficient, exact computation of redressed warped Gabor frames, such that approximate implementations are considered [22].

Finally, it should be noted that the idea of a (non-unitary) logarithmic warping of the frequency axis to obtain wavelet systems from a system of translates was already used in the proof of the so called *painless conditions* for wavelets systems [16]. However, the idea has never been relaxed to other frequency scales so far. While the parallel work by Christensen and Goh [9] focuses on exposing the duality between Gabor and wavelet systems via the mentioned logarithmic warping, we will allow for more general warping functions to generate time-frequency transformations beyond wavelet and Gabor systems. The proposed warping procedure has already proven useful in the area of graph signal processing [57].

2. Preliminaries

We use the following normalization of the Fourier transform $\hat{f}(\xi) := \mathcal{F}f(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi it\xi}\,dt$, for all $f \in \mathbf{L}^1(\mathbb{R})$ and its unitary extension to $\mathbf{L}^2(\mathbb{R})$. The inverse Fourier transform is denoted by $\check{f} = \mathcal{F}^{-1}f$. For an open interval $D \subset \mathbb{R}$, typically $D = \mathbb{R}$ or $D = \mathbb{R}^+$, we use the slightly unusual convention that $\mathbf{L}^2(D) := \{f \in \mathbf{L}^2(\mathbb{R})\colon f(t) = 0$ for almost every $t \in \mathbb{R} \setminus D\}$, such that the Fourier transform and its inverse restrict naturally to $\mathbf{L}^2(D)$. Note that this space is canonically isomorphic to the usual space of square-integrable functions $f\colon D \to \mathbb{R}$. Following this convention, we denote by $\mathbf{L}^{2,\mathcal{F}}(D)$ the space of functions whose Fourier spectrum is restricted to D and

$$\mathbf{L}^{2,\mathcal{F}}(D) := \mathcal{F}^{-1}\left(\mathbf{L}^2(D)\right) \subseteq \mathbf{L}^2(\mathbb{R}).$$

Further, we require the *modulation operator* and the *translation operator* defined by $\mathbf{M}_\omega f =$

$f \cdot e^{2\pi i\omega(\cdot)}$ and $\mathbf{T}_x f = f(\cdot - x)$ respectively for all $f \in \mathbf{L}^2(\mathbb{R})$. The composition $f(g(\cdot))$ of two functions f and g is denoted by $f \circ g$ and the standard Lebesgue measure by μ.

When discussing the properties of the constructed function systems in the following sections, we will repeatedly use the notions of weight functions and weighted $\mathbf{L}^p$-spaces, $1 \leq p \leq \infty$. Weighted $\mathbf{L}^p$-spaces are defined as

$$\mathbf{L}^p_w(\mathbb{R}) := \{f : \mathbb{R} \mapsto \mathbb{C} \colon wf \in \mathbf{L}^p(\mathbb{R})\}.$$

with a continuous, nonnegative function $w : \mathbb{R} \mapsto [0, \infty)$ called *weight function*. The associated norm is $\|f\|_{\mathbf{L}^p_w} := \|wf\|_{\mathbf{L}^p}$. In the following, when the term weight function is used, continuity and non-negativity are always implied.

Two special classes of weight functions are of particular interest: *Continuous, positive weight functions* $v : \mathbb{R} \to \mathbb{R}^+$ and $w : \mathbb{R} \to \mathbb{R}^+$ are called *submultiplicative* and *v-moderate* respectively if they satisfy

$$v(x + y) \leq v(x)v(y), \text{ and } w(x + y) \leq Cv(x)w(y), \tag{1}$$

for all $x, y \in \mathbb{R}$ and some positive constant C. In particular, we can (and will) always choose v such that 1 is a valid choice for the constant in the latter inequality ($\max\{C, 1\}v$ is submultiplicative whenever v is). Submultiplicative and moderate weight functions play an important role in the theory of function spaces, as they are closely related to the translation-invariance of the corresponding weighted spaces [25, 34], see also [35] for an in-depth analysis of weight functions and their role in harmonic analysis.

A generalized shift-invariant (GSI) system on $\mathbf{L}^2(\mathbb{R})$ is a union of shift-invariant systems $\{\mathbf{T}_{na_m} h_m \in \mathbf{L}^2(\mathbb{R}) \colon n \in \mathbb{Z}\}$, with $h_m \in \mathbf{L}^2(\mathbb{R})$ and $a_m \in \mathbb{R}^+$, for all m in some index set. The representation coefficients of a function $f \in \mathbf{L}^2(\mathbb{R})$ with respect to the GSI system are given by the inner products

$$c_f(n, m) := \langle f, \mathbf{T}_{na_m} h_m \rangle = \left(f * \overline{h_m(-\cdot)} \right)(na_m),$$

for all n, m. The above representation of the coefficients in terms of a convolution alludes to the fact that $c_f(\cdot, m)$ is a filtered, and sampled, version of f. This relation justifies our use of filter bank terminology when discussing GSI systems.

Definition 2.1. Let $(g_m)_{m\in\mathbb{Z}} \subset \mathbf{L}^2(D)$ and $(a_m)_{m\in\mathbb{Z}} \subset \mathbb{R}^+$. We call the system

$$(g_{m,n})_{m,n\in\mathbb{Z}}, \quad g_{m,n} := \mathbf{T}_{na_m} \mathcal{F}^{-1}(g_m), \text{ for all } n, m \in \mathbb{Z}, \tag{2}$$

a *(non-uniform) filter bank* for $\mathbf{L}^{2,\mathcal{F}}(D)$. The elements of $(g_m)_{m\in\mathbb{Z}}$ and $(a_m)_{m\in\mathbb{Z}}$ are called *frequency responses* and *decimation factors*, respectively.

Such filter banks can be used to analyze signals in $\mathbf{L}^{2,\mathcal{F}}(D)$, and for a given signal $f \in \mathbf{L}^{2,\mathcal{F}}(D)$, we refer to the sequence $c_f := (c_f(n, m))_{n,m\in\mathbb{Z}} = (\langle f, g_{m,n} \rangle)_{m,n\in\mathbb{Z}}$ as the *filter bank (analysis) coefficients*. A *uniform filter bank* is a filter bank with $a_m = a$ for all $m \in \mathbb{Z}$.

For many applications it is of great importance that all the considered signals can be reconstructed from these coefficients, in a stable fashion. It is a central observation of frame theory that this is equivalent to the existence of constants $0 < A \leq B < \infty$, such that

$$A\|f\|_2^2 \leq \|c_f\|^2_{\ell^2(\mathbb{Z}^2)} \leq B\|f\|_2^2, \text{ for all } f \in \mathbf{L}^{2,\mathcal{F}}(D). \tag{3}$$

A system $(g_{m,n})_{m,n\in\mathbb{Z}}$ that satisfies this condition is called *filter bank frame* [20, 7], and *tight (filter bank) frame* if equality can be achieved in (3). If $(g_{m,n})_{m,n\in\mathbb{Z}}$ is a frame, then the frame operator given by

$$\mathbf{S} : \mathbf{L}^{2,\mathcal{F}}(D) \to \mathbf{L}^{2,\mathcal{F}}(D), \quad \mathbf{S}f = \sum_{m,n\in\mathbb{Z}} c_f(m,n) g_{m,n}, \text{ for all } f \in \mathbf{L}^{2,\mathcal{F}}(D), \tag{4}$$

is invertible. The frame operator is tremendously important and the key component for an appropriate synthesis system that maps the coefficient space $\ell^2(\mathbb{Z}^2)$ to the signal space $\mathbf{L}^{2,\mathcal{F}}(D)$. Namely, the *canonical dual frame* $(\widetilde{g_{m,n}})_{m,n\in\mathbb{Z}}$, obtained by applying the inverse of the frame operator to the frame elements, i.e. $\widetilde{g_{m,n}} := \mathbf{S}^{-1}(g_{m,n})$, for all $m, n \in \mathbb{Z}$, facilitates *perfect reconstruction* from the analysis coefficients:

$$f = \sum_{m,n\in\mathbb{Z}} c_f(m,n)\widetilde{g_{m,n}}, \text{ for all } f \in \mathbf{L}^{2,\mathcal{F}}(D). \tag{5}$$

If at least the upper inequality in (3) is satisfied, then $(g_{m,n})_{m,n\in\mathbb{Z}}$ is a *Bessel sequence*. Note that, in contrast to short-time Fourier or uniform filter bank frames, there is no guarantee that the canonical dual frame, or indeed any dual frame, of a general filter bank frame is of the form $(\mathbf{T}_{na_m}\mathcal{F}^{-1}(\widetilde{g_m}))_{n,m\in\mathbb{Z}}$, for some $(\widetilde{g_m})_{m\in\mathbb{Z}} \subset \mathbf{L}^2(D)$ and the same sequence of decimation factors $(a_m)_{m\in\mathbb{Z}}$. Abstract filter bank frames [6] have received considerable attention, as (generalized) shift-invariant systems in [42, 37, 56, 10, 41] and as (frequency-side) nonstationary Gabor systems in [3, 19, 18, 38]. In contrast, this contribution is concerned with a specific, structured family of filter bank systems and how the superimposed structure can be used to construct filter bank frames.

3. Warped filter banks

In signal analysis, the usage of different frequency scales has a long history. Linear and logarithmic scales arise naturally when constructing a filter bank through modulation or dilation of a single prototype filter, respectively. In this way, the classical Gabor and wavelet transforms are obtained. The consideration of alternative frequency scales can be motivated, for example, from (a) theoretical interest in a family of time-frequency representations that serve as an interpolation between the two extremes, as is the case for the α-transform (which can be related to polynomial scales), or (b) specific applications and/or signal classes. A prime example for the second case is audio signal processing with respect to an auditory frequency scale, e.g. in gammatone filter banks [52, 59] adapted to the ERB scale [33], the latter modeling the frequency progression and frequency-bandwidth relationship in the human cochlea. The mentioned methods have in common that they are based on a single prototype filter and possess the structure of a GSI (or filter bank) system and that the bandwidth of the filters is directly linked to the filter center frequencies and their spacing.

The filter banks we propose in this section have the property that they are designed as a system of translates on a given frequency scale. This scale determines a conversion from frequency to a new unit (e.g. ERB) with respect to which the designed filters provide a uniform resolution. In the next sections, we will show that this construction admits a special class of non-uniform filter banks with a simplified structure compared to general filter banks.

Formally, a frequency scale is specified by a continuous, bijective function $\Phi : D \to \mathbb{R}$ and the transition between the non-linear scale Φ and the unit linear scale is achieved by Φ and Φ^{-1}. Hence, we construct filter frequency responses from a prototype function $\theta : \mathbb{R} \mapsto \mathbb{C}$ by means of *translation*, followed by *deformation*,

$$((\mathbf{T}_m\theta)\circ\Phi)_{m\in\mathbb{Z}}\,. \tag{6}$$

This general formulation provides tremendous flexibility for frequency scale design. Furthermore, choosing Φ as $\Phi(\xi)\mapsto a\xi$ or $\Phi(\xi)\mapsto \log_a(\xi)$, for $a>0$, yields systems of translates $\mathbf{T}_{m/a}\left(\theta(a\cdot)\right)$ and dilates $(\theta\circ\log_a)(\cdot/a^m)$, respectively. Such Φ will provide the starting point for recovering Gabor and wavelet filter banks in our framework.

Definition 3.1. Let $D\subseteq\mathbb{R}$ be any open interval. A $\mathcal{C}^1$-diffeomorphism $\Phi: D\to\mathbb{R}$ is called *warping function*, if

(i) the derivative Φ' of Φ is positive, i.e. $\Phi'>0$, and
(ii) there is a submultiplicative weight v, such that the weight function

$$w := \left(\Phi^{-1}\right)' = \frac{1}{\Phi'\left(\Phi^{-1}(\cdot)\right)} \tag{7}$$

is v-moderate, i.e. $w(\tau_0+\tau_1)\le v(\tau_0)w(\tau_1)$, for all $\tau_0,\tau_1\in\mathbb{R}$.

Given a warping function Φ, w and v will from now on always denote weights as specified in Definition 3.1.

Remark 3.2. While moderateness and invertibility of Φ will prove essential for our results, there are no technical obstructions preventing us from allowing warping functions $\Phi\in\mathcal{C}^0(D)\setminus\mathcal{C}^1(D)$, such that Φ' is only piecewise continuous. However, this implies that some (or all) of the elements of the warped family given in (6) can have at most piecewise continuous derivative, independent of the smoothness of θ and with the implied negative effects to their Fourier localization. Moreover, Φ is easily lifted to $\mathcal{C}^1$ with minor, arbitrarily local changes. Therefore, we only see limited value in generalizing the notion of a warping function beyond diffeomorphisms.

Proposition 3.3. *If $\Phi: D\to\mathbb{R}$ is a warping function as per Definition 3.1, then $\tilde{\Phi}:=c\Phi(\cdot/d)$ is a warping function with domain dD, for all positive, finite constants $c,d\in\mathbb{R}^+$. If $w=(\Phi^{-1})'$ is v-moderate, then $\tilde{w}=(\tilde{\Phi}^{-1})'$ is $v(\cdot/c)$-moderate.*

Proof. The result is easily obtained by elementary manipulation. □

Several things should be noted when considering the definition and proposition above.

- Proposition 3.3 shows that it really is sufficient to consider integer translates of the prototype θ when constructing the frequency responses $\theta_{\Phi,m}$. If $a>0$ is arbitrary, then with $\theta_a:=\theta(\cdot/a)$, we have

$$(\mathbf{T}_m\theta_a)\circ(a\Phi)=\theta_a\left(a\Phi(\cdot)-m\right)=\theta\left(\Phi(\cdot)-m/a\right)=(\mathbf{T}_{m/a}\theta)\circ\Phi. \tag{8}$$

- Moderateness of $w=\left(\Phi^{-1}\right)'$ ensures translation invariance of the associated weighted $\mathbf{L}^p$-spaces. In particular, identifying $(\mathbf{T}_m\theta)\circ\Phi$ with its trivial extension to the whole real line, we have

$$\|(\mathbf{T}_m\theta)\circ\Phi\|^2_{\mathbf{L}^2(D)}=\|\mathbf{T}_m\theta\|^2_{\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})}\le\begin{cases} v(m)\|\theta\|^2_{\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})} & \text{, if } \theta\in\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})\\ w(m)\|\theta\|^2_{\mathbf{L}^2_{\sqrt{v}}(\mathbb{R})} & \text{, if } \theta\in\mathbf{L}^2_{\sqrt{v}}(\mathbb{R}).\end{cases} \tag{9}$$

- $\mathbf{L}^2_{\sqrt{v}}(\mathbb{R}) \subseteq \mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$, since (9), with $m = 0$, implies $\|\theta\|^2_{\mathbf{L}^2_{\sqrt{w}}(\mathbb{R})} \leq w(0)\|\theta\|^2_{\mathbf{L}^2_{\sqrt{v}}(\mathbb{R})}$.

A warped filter bank can now be constructed easily. To do so, after selecting the warping function Φ, one simply chooses an appropriate prototype frequency response θ and positive, real decimation factors $(a_m)_{m\in\mathbb{Z}}$. Although, in theory, the choice of decimation factors is arbitrary, the warping function Φ induces a canonical choice, which relates a_m^{-1} to the essential support of the frequency responses and is particularly suited for the creation of warped filter bank frames, see Section 4.

Definition 3.4. Let $\Phi : D \to \mathbb{R}$ be a warping function and $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$. Furthermore, let $\mathbf{a} := (a_m)_{m\in\mathbb{Z}} \subset \mathbb{R}^+$ be a set of decimation factors. Then the *warped filter bank* with respect to the triple $(\Phi, \theta, \mathbf{a})$ is given by

$$\mathcal{G}(\Phi, \theta, \mathbf{a}) := (\mathbf{T}_{na_m}\widecheck{g_m})_{m,n\in\mathbb{Z}} = \left(\mathbf{T}_{na_m}\mathcal{F}^{-1}(g_m)\right)_{m,n\in\mathbb{Z}}, \tag{10}$$

with

$$g_m(\xi) := \begin{cases} \sqrt{a_m}(\mathbf{T}_m\theta) \circ \Phi(\xi) & \text{if } \xi \in D, \\ 0 & \text{else.} \end{cases} \tag{11}$$

If $a_m = \tilde{a}/w(m)$, for all $m \in \mathbb{Z}$ and some $\tilde{a} > 0$, then we say that $\mathbf{a}$ is a set of *natural decimation factors (for* (Φ, θ)*).*

Natural decimation factors are very important, as they guarantee uniform $\mathbf{L}^2$-boundedness of the g_m, recall (9), which is in turn easily seen to be necessary for the Bessel property.

Remark 3.5. Note that the condition $\theta \in \mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$ would be sufficient to ensure that $\mathcal{G}(\Phi, \theta, \mathbf{a}) \subset \mathbf{L}^{2,\mathcal{F}}(D)$. In that setting, a set of natural downsampling factors would have the form $a_m = \tilde{a}/v(m)$ for all $m \in \mathbb{Z}$ and some $\tilde{a} > 0$, instead. All results in this contribution also hold in this case and are proven with the same techniques. Since a decay condition on θ is usually considered less severe than a restriction of the sampling density, our results are presented for the configuration given in Definition 3.4.

However, depending on how much the submultiplicative weight v deviates from $w = (\Phi^{-1})'$, the two sets of natural decimation factors, and the spaces of eligible prototype functions, may differ significantly. Therefore, we shortly discuss the necessary changes in the case $\theta \notin \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$ in Section 4.1.

Assume for now that $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R}) \cap \mathbf{L}^1_w(\mathbb{R})$. If we rewrite the elements of $\mathcal{G}(\Phi, \theta, \mathbf{a})$ as a Fourier integral, i.e.

$$\begin{aligned}\mathcal{F}^{-1}(g_m)(t) &= \int_{\mathbb{R}} g_m(\xi)e^{2\pi i\xi t}\, d\xi \\ &= \sqrt{a_m}\int_D \theta(\Phi(\xi) - m)e^{2\pi i\xi t}\, d\xi = \sqrt{a_m}\int_{\mathbb{R}} w(\tau + m)\theta(\tau)e^{2\pi i\Phi^{-1}(\tau+m)t}\, d\tau,\end{aligned}$$

with the change of variable $\xi = \Phi^{-1}(\tau + m)$, we can see that decay (smoothness) of θ implies smoothness (decay) for the elements of $\mathcal{G}(\Phi, \theta, \mathbf{a})$, provided that Φ is smooth enough as well. This behavior is crucial for the construction of systems with good time-frequency localization and in fact central for the results presented in [40], where the above Fourier integrals are studied in more detail.

We now provide some examples of warping functions that are of particular interest, e.g. because they encompass important frequency scales. In Proposition 3.11 at the end of this section, we show that the presented examples indeed define warping functions in the sense of Definition 3.1. Some instances of the warping functions in the following examples can be seen in Figure 1.

Example 3.6 (Wavelets). Choosing $\Phi = \log$, with $D = \mathbb{R}^+$ leads to a system of the form

$$g_m(\xi) = \sqrt{a_m}\theta(\log(\xi) - m) = \sqrt{a_m}\theta(\log(\xi e^{-m})) = \sqrt{\frac{a_m}{a_0}} g_0(\xi e^{-m}).$$

This warping function therefore leads to g_m being a dilated version of some $g_0 = \theta \circ \log$. The natural decimation factors are given by $a_m = \tilde{a}/w(m) = \tilde{a}e^{-m}$. This shows that $\mathcal{G}(\log, \theta, \tilde{a}e^{-m})$ is indeed a wavelet system, with the minor modification that our scales are reciprocal to the usual definition of wavelets.

Example 3.7. The family of warping functions $\Phi_l(\xi) = c\left((\xi/d)^l - (\xi/d)^{-l}\right)$, for some $c, d > 0$ and $l \in (0, 1]$, is an alternative to the logarithmic warping for the domain $D = \mathbb{R}^+$. The logarithmic warping in the previous example can be interpreted as the limit of this family for $l \to 0$ in the sense that for any fixed $\xi \in \mathbb{R}^+$,

$$\Phi_l'(\xi)/l = \frac{c}{d}\left((\xi/d)^{-1+l} + (\xi/d)^{-1-l}\right) \overset{l \to 0}{\to} \frac{2c}{\xi} = \frac{2c}{d}\log'(\xi/d).$$

This type of warping provides a frequency scale that approaches the limits 0 and ∞ of the frequency range D in a slower fashion than the wavelet warping. In other words, g_m is less deformed for $m > 0$, but more deformed for $m < 0$ than in the case $\Phi = \log$. Furthermore, the property that g_m can be expressed as a dilated version of g_0 is lost.

Example 3.8 (ERBlets). In psychoacoustics, the investigation of filter banks adapted to the spectral resolution of the human ear has been subject to a wealth of research, see [46] for an overview. We mention here the Equivalent Rectangular Bandwidth scale (ERB-scale) described in [33], which introduces a set of bandpass filters modeling human perception, see also [48] for the construction of an invertible filter bank adapted to the ERB-scale. In our terminology the ERB warping function is given by

$$\Phi_{\mathrm{ERB}}(\xi) = \operatorname{sgn}(\xi)\, c \log\left(1 + \frac{|\xi|}{d}\right),$$

where the constants are given by $c = 9.265$ and $d = 228.8$. Using this function, we obtain a filter bank with an equivalent time-frequency resolution trade-off to the construction in [48] or more traditional Gammatone filter banks on the ERB-scale, see e.g. [59]. However, we will see in Section 4 that it is very easy to construct tight and snug warped filter banks, while at least tightness is usually not achievable by these previous constructions. The ERB filter bank has potential applications in audio signal processing, as it provides a perfectly invertible transform adapted to the human perception of sound.

Example 3.9. Filter banks obtained from the warping functions $\Phi_\alpha(\xi) = \operatorname{sgn}(\xi)\left((|\xi| + 1)^{1-\alpha} - 1\right)$, for some $\alpha \in [0, 1)$ can serve as a substitute for the α-*transform*, see [12, 31, 47, 26]. The latter is a filter bank constructed from a single prototype frequency response by translation and dilation, leading to filters with bandwidth proportional to $(1 + |\xi|)^\alpha$, where $\xi \in \mathbb{R}$ is the center frequency of the filter frequency response. Varying α, one can *interpolate* between the Gabor transform ($\alpha = 0$, constant time-frequency resolution) and a wavelet-like (or more precisely ERB-like) transform with the dilation depending linearly on the center frequency ($\alpha \to 1$). It is easy to confirm that the warping function $\Phi_\alpha(\xi)$ yields the same qualitative center frequency to bandwidth relationship. We will see in subsequent sections that, in stark contrast to the α-transform, it is easy to construct tight frames using the warping function Φ_α. Note as well that study of the α-transform usually excludes the limiting case $\alpha = 1$, which is also not captured by the above warping construction. However, the logarithmic warping considered in Example 3.8 yields filters with bandwidth proportional to $(1 + |\xi|)$ and can thus be considered as substitute for the limiting case.

Example 3.10. Finally, we propose a warping function for representing functions band-limited to the interval $D = (-\pi, \pi)$. For this purpose set $\Phi(\xi) = \tan(\xi)$. Necessarily, the frequency responses g_m, given by (11), are all compactly supported on D and increasingly peaky and concentrated at the upper and lower borders of D, as m tends to ∞ and $-\infty$, respectively. By using the equivalence of GSI systems and nonstationary Gabor systems [3] through application of the Fourier transform, we can thus construct time-frequency systems on arbitrary open intervals. Frames for intervals have been proposed previously by Abreu et al. [1].

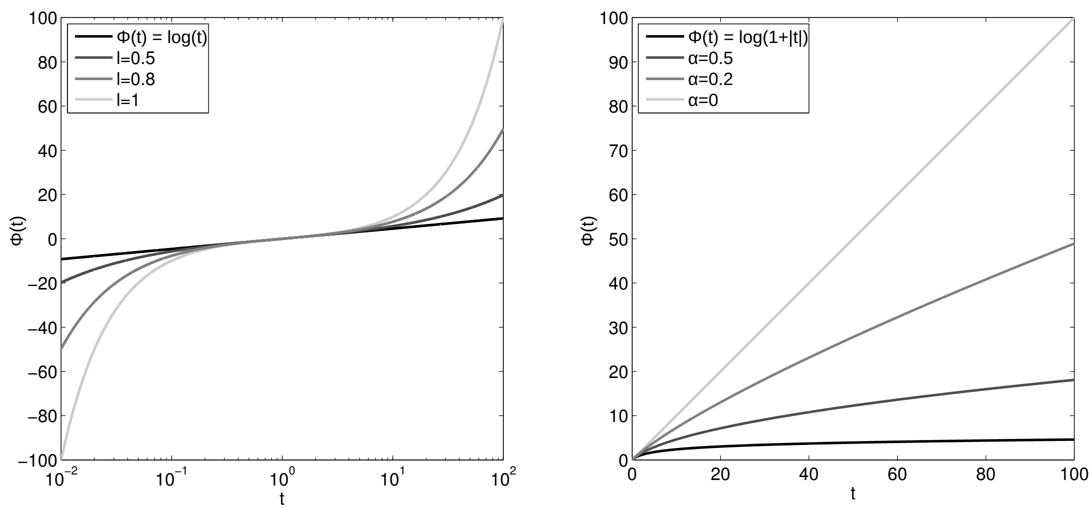


Figure 1: (left) Warping functions from Examples 3.6 and 3.7: This plot shows logarithmic (wavelet) warping function (black) and $\Phi_l = l^{-1}(\xi^l - \xi^{-l})$, for $l = 0.5$ (dark gray), $l = 0.8$ (medium gray) and $l = 1$ (light gray). Note that the horizontal axis is logarithmic. (right) Warping functions from Examples 3.8 and 3.9. This plot shows the ERBlet warping function (black), with $c = d = 1$, and $\Phi_\alpha = (1-\alpha)^{-1}\,\mathrm{sgn}(\xi)((1 + |\xi|)^{(1-\alpha)} - 1)$, for $\alpha = 0.5$ (dark gray), $\alpha = 0.2$ (medium gray) and $\alpha = 0$ (light gray). The horizontal axis is linear.

Proposition 3.11. *The following are valid triples of warping functions, weights w and moderating submultiplicative weights v, as per Definition 3.1:*

(i) $\Phi : \mathbb{R}^+ \to \mathbb{R},\ \xi \mapsto \log(\xi)$, *with* $w = v = e^{(\cdot)}$.

(ii) $\Phi : \mathbb{R}^+ \to \mathbb{R},\ \xi \mapsto \big((\xi)^l - (\xi)^{-l}\big)$, *for* $l \in (0, 1]$, *with*

$$w = l^{-1}\left(\left(\frac{2}{(\cdot) + \sqrt{(\cdot)^2 + 4}}\right)^{(l-1)/l} + \left(\frac{(\cdot) + \sqrt{(\cdot)^2 + 4}}{2}\right)^{(l+1)/l}\right) \quad \text{and} \quad v = (2+|\cdot|/2)^{(2+2l)/l}.$$

(iii) $\Phi : \mathbb{R} \to \mathbb{R},\ \xi \mapsto \operatorname{sgn}(\xi)\ \log(1+|\xi|)$ *with* $w = v = e^{|\cdot|}$.

(iv) $\Phi : \mathbb{R} \to \mathbb{R},\ \xi \mapsto \operatorname{sgn}(\xi)\ \big((1+|\xi|)^{1-\alpha} - 1\big)$, *for some* $\alpha \in [0,1)$, *with* $w = v/(1-\alpha) = (1-\alpha)^{-1}(1+|\cdot|)^{\alpha/(1-\alpha)}$.

(v) $\Phi : (-\pi,\pi) \to \mathbb{R},\ \xi \mapsto \tan(\xi)$, *with* $w = (1+(\cdot)^2)^{-1}$ *and* $v = 2(1+(\cdot)^2)$.

Proof. Items (i), (iii) and (iv) are easily shown through elementary calculations. It remains to prove items (ii) and (v).

Ad (ii): Φ is in $\mathcal{C}^\infty(\mathbb{R}^+)$ and

$$w(\tau_0) = \left(\Phi^{-1}\right)'(\tau_0) = \frac{1}{\Phi'(\Phi^{-1}(\tau_0))} = l^{-1}\frac{\Phi^{-1}(\tau_0)}{\Phi^{-1}(\tau_0)^l + \Phi^{-1}(\tau_0)^{-l}}.$$

Assume that Φ^{-1} is $\tilde{v}$-moderate, which also implies $\Phi^{-1}(\tau_0+\tau_1) \geq \Phi^{-1}(\tau_0)\tilde{v}(-\tau_1)^{-1}$, then

$$w(\tau_0+\tau_1) = l^{-1}\frac{\Phi^{-1}(\tau_0+\tau_1)}{\Phi^{-1}(\tau_0+\tau_1)^l + \Phi^{-1}(\tau_0+\tau_1)^{-l}} \leq w(\tau_0)\max\{\tilde{v}(\tau_1),\tilde{v}(-\tau_1)\}^{l+1}.$$

The inverse of Φ is given by $\Phi^{-1}(\tau_0) = 2^{-1/l}(\tau_0 + \sqrt{\tau_0^2+4})^{1/l}$, for all $\tau_0 \in \mathbb{R}$. If we assume that

$$\frac{\tau_1+\tau_0+\sqrt{(\tau_1+\tau_0)^2+4}}{\tau_0+\sqrt{\tau_0^2+4}} = 1 + \frac{\tau_1+\sqrt{(\tau_1+\tau_0)^2+4}-\sqrt{\tau_0^2+4}}{\tau_0+\sqrt{\tau_0^2+4}} \leq v_0(\tau_1), \tag{12}$$

for some submultiplicative function v_0, then Φ^{-1} is $\tilde{v}$-moderate with $\tilde{v} = v_0^{1/l}$. In fact, (12) holds with $v_0 = (2+|\cdot|/2)^2$, such that we obtain v-moderateness of w with $v := v_0^{\frac{l+1}{l}} = (2+|\cdot|/2)^{\frac{2+2l}{l}}$.

To see this, observe that

$$\frac{\tau_1+\sqrt{(\tau_1+\tau_0)^2+4}-\sqrt{\tau_0^2+4}}{\tau_0+\sqrt{\tau_0^2+4}} \leq \begin{cases} 0 & \text{for all } \tau_1 \leq 0, \\ \frac{\tau_1}{\sqrt{\tau_1^2/4+4}-\tau_1/2} & \text{else,} \end{cases}$$

since the left hand side attains its global maximum at $\tau_0 = -\tau_1/2$, for fixed $\tau_1 > 0$. Apply the fundamental theorem of calculus with $f = \sqrt{\tau_1^2/4+(\cdot)}$ to obtain the estimate

$$\frac{\tau_1}{\sqrt{\tau_1^2/4+4}-\tau_1/2} \leq \frac{\tau_1^2+16}{4} \leq (2+|\tau_1|/2)^2,$$

valid for all $\tau_1 > 0$, as desired.

Ad (v): The crucial step is to show that $\arctan' = (1+(\cdot)^2)^{-1}$ can be moderated by a submultiplicative weight. But since $(\tau_0+\tau_1)^2 \leq 2(\tau_0^2+\tau_1^2)$ for all $\tau_0,\tau_1 \in \mathbb{R}$, it is easy to see that $v = 2(1+(\cdot)^2)$ is submultiplicative and that $w = (1+(\cdot)^2)^{-1}$ is v-moderate. The other required properties of $\Phi = \tan$ are elementary. □

4. Warped filter bank frames

Although this contribution is concerned only with warped filter bank frames, our results are derived from structural properties and results obtained in the general, abstract filter bank (or GSI) setting. As such, the structure imposed on warped filter banks can be seen as a constructive means to satisfy, or simplify, the conditions of these abstract results. Our results rely on the simple, but crucial identity

$$\sum_{m\in\mathbb{Z}} a_m^{-1}|g_m(\xi)|^2 = \sum_{m\in\mathbb{Z}} |(\mathbf{T}_m\theta)\circ\Phi(\xi)|^2, \text{ for all } \xi\in D, \tag{13}$$

which holds for every warped filter bank $\mathcal{G}(\Phi,\theta,\mathbf{a})$. As a consequence of the above equality, we can find upper and lower bounds for (13), by instead determining upper and lower bounds on the simpler quantity

$$\sum_{m\in\mathbb{Z}} |\mathbf{T}_m\theta|^2. \tag{14}$$

In order to exclude pathological cases from the study of filter bank frames, it has proven useful to assume that a filter bank $(g_{m,n})_{m,n\in\mathbb{Z}}$ satisfies the so-called local integrability condition [37, 41, 10]. This enables the generalization of numerous important results, e.g. a characterization of dual frames, from the frame theory of Gabor systems [34] and uniform filter banks [42].

Definition 4.1. Denote by $\mathcal{D}$ the set of all functions $f\in\mathbf{L}^\infty(D)$ with compact support. We say that the filter bank $(g_{m,n})_{m,n\in\mathbb{Z}}$, generated from $(g_m)_{m\in\mathbb{Z}}\subset\mathbf{L}^2(D)$ and $(a_m)_{m\in\mathbb{Z}}\subset\mathbb{R}^+$, satisfies the local integrability condition (LIC), if

$$L(f) := \sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} a_m^{-1}\int_{\operatorname{supp}(f)} \left|f(\xi+la_m^{-1})g_m(\xi)\right|^2\ d\xi < \infty, \tag{15}$$

for all $f\in\mathcal{D}$.

The LIC might seem intimidating and opaque at first, but once we impose some structure on $(g_{m,n})_{m,n\in\mathbb{Z}}$, it can often be substituted by mild conditions on the frequency responses g_m and decimation factors a_m. In the case of warped filter banks, boundedness of (14) and $\mathbf{a}$ being majorized by a set of natural decimation factors is already sufficient for $\mathcal{G}(\Phi,\theta,\mathbf{a})$ to satisfy the LIC. Therefore, the results presented in [37, 41, 10], many of which require the LIC, are available to a large class of warped filter banks.

Theorem 4.2. *Let $\Phi: D\to\mathbb{R}$ be a warping function and $\theta\in\mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$. If*

$$\sup_{m\in\mathbb{Z}} a_m w(m) < \infty \qquad \textit{and} \qquad \operatorname*{ess\,sup}_{\tau\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\mathbf{T}_m\theta(\tau)|^2 < \infty, \tag{16}$$

then $\mathcal{G}(\Phi,\theta,\mathbf{a})$ satisfies the LIC (15)*. In particular, if $\mathbf{a}$ is a set of natural decimation factors, then $\mathcal{G}(\Phi,\theta,\mathbf{a})$ satisfies the LIC if the second condition in* (16) *holds.*

Proof. First note that, instead of considering all compactly supported and essentially bounded functions $f\in\mathbf{L}^2(D)$, it is sufficient to verify the LIC (15) only for the characteristic functions $\mathbb{1}_I$ on compact intervals $I\subset D$. These functions are clearly contained in $\mathbf{L}^2(D)$ and it is easy to see that

$$\operatorname{supp}(f)\subseteq I \quad\Longrightarrow\quad L(f)\le \|f\|_\infty^2 L(\mathbb{1}_I).$$

For $\mathbb{1}_I$, the LIC reads

$$L(\mathbb{1}_I) = \sum_{m\in\mathbb{Z}} a_m^{-1}\sum_{l\in\mathbb{Z}}\int_I \mathbb{1}_{I+la_m^{-1}}(\xi)\left|g_m(\xi)\right|^2\ d\xi. \tag{17}$$

If the right hand side of (17) is finite, then it converges absolutely and we can interchange sums and integrals freely. Hence,

$$\begin{aligned}
L(\mathbb{1}_I) &= \sum_{m\in\mathbb{Z}} a_m^{-1} \int_I |g_m(\xi)|^2 \sum_{l\in\mathbb{Z}} \mathbb{1}_{I+la_m^{-1}}(\xi)\ d\xi \\
&< \sum_{m\in\mathbb{Z}} \frac{a_m\mu(I)+1}{a_m} \int_I |g_m(\xi)|^2\ d\xi \\
&= \sum_{m\in\mathbb{Z}} (a_m\mu(I)+1) \int_I |(\mathbf{T}_m\theta)\circ\Phi(\xi)|^2\ d\xi,
\end{aligned}$$

where we used that $\sum_{l\in\mathbb{Z}} \mathbb{1}_{I+la_m^{-1}}(\xi) \le \lceil a_m\mu(I)\rceil < a_m\mu(I)+1$ for arbitrary $\xi \in D$. We split the upper estimate into two terms and interchange integration and summation once more to obtain

$$L(\mathbb{1}_I) < \int_I \sum_{m\in\mathbb{Z}} |(\mathbf{T}_m\theta)\circ\Phi(\xi)|^2\ d\xi + \sum_{m\in\mathbb{Z}} a_m\mu(I)\cdot \int_I |(\mathbf{T}_m\theta)\circ\Phi(\xi)|^2\ d\xi.$$

By assumption, there is some constant $B>0$, such that $\sum_{m\in\mathbb{Z}} |\mathbf{T}_m\theta(\tau)|^2 < B$ almost everywhere and we can conclude that

$$\int_I \sum_{m\in\mathbb{Z}} |(\mathbf{T}_m\theta)\circ\Phi(\xi)|^2\ d\xi \le \mu(I)B.$$

To estimate the second term, note that the change of variable $\xi = \Phi^{-1}(\tau+m)$ yields

$$\sum_{m\in\mathbb{Z}} a_m \cdot \int_I |(\mathbf{T}_m\theta)\circ\Phi(\xi)|^2\ d\xi = \sum_{m\in\mathbb{Z}} a_m \cdot \int_{\Phi(I)-m} w(\tau+m)\,|\theta(\tau)|^2\ d\tau = (*).$$

By assumption $\sup_{m\in\mathbb{Z}} a_m w(m) < \infty$ and $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$. To estimate the right hand side of the above equation, we can use v-moderateness of w:

$$\begin{aligned}
(*) &\le \sum_{m\in\mathbb{Z}} a_m w(m) \cdot \int_{\Phi(I)-m} v(\tau)\,|\theta(\tau)|^2\ d\tau \\
&= \sum_{m\in\mathbb{Z}} a_m w(m) \cdot \int_{\mathbb{R}} \mathbb{1}_{\Phi(I)-m} v(\tau)\,|\theta(\tau)|^2\ d\tau \\
&= \sup_{m\in\mathbb{Z}} a_m w(m) \cdot \int_{\mathbb{R}} v(\tau)\,|\theta(\tau)|^2 \sum_{m\in\mathbb{Z}} \mathbb{1}_{\Phi(I)-m}\ d\tau \\
&< (1+\mu(\Phi(I)))\cdot \sup_{m\in\mathbb{Z}} a_m w(m) \cdot \int_{\mathbb{R}} v(\tau)\,|\theta(\tau)|^2\ d\tau \\
&= (1+\mu(\Phi(I)))\cdot \sup_{m\in\mathbb{Z}} a_m w(m) \cdot \|\theta\|^2_{\mathbf{L}^2_{\sqrt{v}}} < \infty.
\end{aligned}$$

Altogether, we obtain

$$L(\mathbb{1}_I) < \mu(I)\cdot\left(B + (1+\mu(\Phi(I)))\cdot \sup_{m\in\mathbb{Z}} a_m w(m)\cdot \|\theta\|^2_{\mathbf{L}^2_{\sqrt{v}}}\right) < \infty,$$

which establishes the desired result. If $\mathbf{a}$ is a set of natural decimation factors, then $a_m w(m) = \tilde{a} < \infty$ for all $m \in \mathbb{Z}$, yielding the second claim. □

Note that the second condition in (16) is in fact necessary for $\mathcal{G}(\Phi, \theta, \mathbf{a})$ to be a Bessel sequence, as can be seen in the next result, where we provide necessary and sufficient conditions for a warped filter bank to form a Bessel sequence or even a frame. Proving these conditions amounts to combining previous results from the theory of filter bank frames, in particular from [10, 3, 38], with the special structure of warped filter banks.

Theorem 4.3. *Let $\mathcal{G}(\Phi, \theta, \mathbf{a})$ be a warped filter bank for $\mathbf{L}^{2,\mathcal{F}}(D)$.*

(i) If $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a Bessel sequence with bound $B < \infty$, then

$$\sum_{m \in \mathbb{Z}} |\mathbf{T}_m \theta(\tau)|^2 \leq B < \infty, \text{ for almost all } \tau \in \mathbb{R}. \tag{18}$$

If $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a frame with lower bound $A > 0$ and $\sup_{m \in \mathbb{Z}} a_m w(m) < \infty$, then

$$0 < A \leq \sum_{m \in \mathbb{Z}} |\mathbf{T}_m \theta(\tau)|^2, \text{ for almost all } \tau \in \mathbb{R}. \tag{19}$$

(ii) Assume that, there are constants $c < d$, such that $\operatorname{supp}(\theta) \subseteq [c, d]$. If $a_m^{-1} \geq \Phi^{-1}(d+m) - \Phi^{-1}(c+m)$, for all $m \in \mathbb{Z}$, then $\mathcal{G}(\Phi, \theta, \mathbf{a})$ forms a frame for $\mathbf{L}^{2,\mathcal{F}}(D)$, with frame bounds A, B, if and only if $0 < A \leq \sum_{m \in \mathbb{Z}} |\mathbf{T}_m \theta|^2 \leq B < \infty$ almost everywhere. Furthermore, the canonical dual frame for $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is given by $\mathcal{G}(\Phi, \tilde{\theta}, \mathbf{a})$, with

$$\tilde{\theta} = \frac{\theta}{\sum_{l \in \mathbb{Z}} |\mathbf{T}_l \theta|^2}. \tag{20}$$

Proof. To prove the first part of (i) apply [38, Proposition 3], which states that if $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a Bessel sequence with bound $B < \infty$, then $\sum_{m \in \mathbb{Z}} a_m^{-1} |g_m(\xi)|^2 \leq B$, for almost all $\xi \in D$. However, by (13), this is equivalent to $\sum_{m \in \mathbb{Z}} |\mathbf{T}_m \theta|^2 \leq B$ almost everywhere on $\mathbb{R}$. For the second part, first note that the frame property implies the Bessel property, such that we obtain $\operatorname{ess\,sup}_{\tau \in \mathbb{R}} \sum_{m \in \mathbb{Z}} |\mathbf{T}_m \theta(\tau)|^2 < \infty$. Since $\sup_{m \in \mathbb{Z}} a_m w(m) < \infty$, we can apply Theorem 4.2 to ensure that $\mathcal{G}(\Phi, \theta, \mathbf{a})$ satisfies the LIC. Therefore, we can apply [10, Corollary 3.4], which states that if $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a frame with lower frame bound $A > 0$ and $\mathcal{G}(\Phi, \theta, \mathbf{a})$ satisfies the LIC, then $A \leq \sum_{m \in \mathbb{Z}} a_m^{-1} |g_m(\xi)|^2$, for almost all $\xi \in D$. Through (13), this implies (19), finishing the proof of (i).

To prove (ii), begin by noting that $\operatorname{supp}(\theta) \subseteq [c, d]$ implies $\operatorname{supp}(g_m) \subseteq [\Phi^{-1}(c+m), \Phi^{-1}(d+m)] \subset D$, for all $m \in \mathbb{Z}$. Hence, we can apply [3, Corollary 1]: If $a_m^{-1} \geq \Phi^{-1}(d+m) - \Phi^{-1}(c+m)$ then $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a frame for for $\mathbf{L}^{2,\mathcal{F}}(D)$, with frame bounds $0 < A \leq B < \infty$, if and only if

$$0 < A \leq \sum_{m \in \mathbb{Z}} a_m^{-1} |g_m(\xi)|^2 \leq B < \infty, \text{ for almost all } \xi \in D. \tag{21}$$

Using (13) once more, this is equivalent to $0 < A \leq \sum_{m \in \mathbb{Z}} |\mathbf{T}_m \theta|^2 \leq B < \infty$ almost everywhere as desired. Moreover, [3, Corollary 1] states that the canonical dual frame $(\widetilde{g_{m,n}})_{m,n \in \mathbb{Z}} \subset \mathbf{L}^{2,\mathcal{F}}(D)$

has the form $\widetilde{g_{m,n}} = \mathbf{T}_{na_m}\mathcal{F}^{-1}(\widetilde{g_m})$, with

$$\widetilde{g_m}(\xi) = \frac{g_m(\xi)}{\sum_{l\in\mathbb{Z}} a_l^{-1}|g_l(\xi)|^2}, \text{ for all } m \in \mathbb{Z} \text{ and almost every } \xi \in D. \tag{22}$$

Inserting (13), respectively the definition of the g_m, we obtain

$$\frac{g_m(\xi)}{\sum_{l\in\mathbb{Z}} a_l^{-1}|g_l(\xi)|^2} = \sqrt{a_m}\frac{(\mathbf{T}_m\theta)\circ\Phi(\xi)}{\sum_{l\in\mathbb{Z}}|(\mathbf{T}_l\theta)\circ\Phi(\xi)|^2} = \sqrt{a_m}\left(\frac{\mathbf{T}_m\theta}{\sum_{l\in\mathbb{Z}}|\mathbf{T}_l\theta|^2}\right)\circ\Phi(\xi).$$

Now simply note that $\sum_{l\in\mathbb{Z}}|\mathbf{T}_l\theta|^2$ is 1-periodic to see that

$$\frac{\mathbf{T}_m\theta}{\sum_{l\in\mathbb{Z}}|\mathbf{T}_l\theta|^2} = \mathbf{T}_m\left(\frac{\theta}{\sum_{l\in\mathbb{Z}}|\mathbf{T}_l\theta|^2}\right).$$

Hence, with $\tilde{\theta}$ as in (20), and trivially extending $\widetilde{g_m}$ from D to $\mathbb{R}$, we obtain $(\widetilde{g_{m,n}})_{m,n\in\mathbb{Z}} = \mathcal{G}(\Phi,\tilde{\theta},\mathbf{a})$ as desired. □

Note that the canonical dual frame in Theorem 4.3(ii) only differs from $\mathcal{G}(\Phi,\theta,\mathbf{a})$ by the choice of prototype filter $\tilde{\theta}$, which is easily computed using Eq (20). Theorem 4.3(ii) is the natural generalization of the classical *painless nonorthogonal expansions* [16] to warped filter banks and extremely useful when strictly bandlimited filters are required. The whole of Theorem 4.3 serves as a strong indicator that for any *snug* frame, i.e. with $B/A \approx 1$, the sum $\sum_{m\in\mathbb{Z}}|\mathbf{T}_m\theta|^2$ must necessarily be close to constant, i.e. it is imperative that the translates of the original window θ have good summation properties. Hence, there is an intimate relationship between stability of the filter bank $\mathcal{G}(\Phi,\theta,\mathbf{a})$ and the shape of (13).

Sometimes, when we would like to work in the setting of Theorem 4.3(ii) it can be more efficient to estimate the support of the g_m instead of calculating it exactly. The following result and its discussion below show that this can easily be done using natural decimation factors, which often provide close-to-optimal values for the a_m that satisfy the conditions of Theorem 4.3(ii).

For the purpose of the following result and for later use, we define the function $V:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$V(\tau_0,\tau_1) := \int_{\tau_0}^{\tau_1} v(\tau)\ d\tau, \text{ for all } \tau_0,\tau_1 \in \mathbb{R}. \tag{23}$$

Corollary 4.4. *Let $\mathcal{G}(\Phi,\theta,\mathbf{a})$ be a warped filter bank with compactly supported prototype $\theta \in \mathbf{L}^2(\mathbb{R})$. Furthermore, define*

$$c_0 := \inf\operatorname{supp}(\theta) \quad \textit{and} \quad d_0 := \sup\operatorname{supp}(\theta)$$

and let $\widetilde{a_w} := V(c_0,d_0)^{-1}$. If

$$a_m \leq \widetilde{a_w}/w(m),$$

then $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a frame with frame bounds A,B, if and only if $0 < A \leq \sum_{m\in\mathbb{Z}}|\mathbf{T}_m\theta|^2 \leq B < \infty$ almost everywhere. In that case, the canonical dual frame is given by $\mathcal{G}(\Phi,\tilde{\theta},\mathbf{a})$, with $\tilde{\theta}$ as in (20).

Note that $V(c_0,d_0)$ can be bounded from above by $(d_0 - c_0)\max_{\tau\in[c_0,d_0]} v(\tau)$. This coarser estimate can be used for an even simpler computation of decimation factors appropriate for Corollary 4.4, e.g. if v is nondecreasing away from zero.

Proof of Corollary 4.4. All we need to do is to invoke the fundamental theorem of calculus to show that $\Phi^{-1}(d_0+m) - \Phi^{-1}(c_0+m) \leq a_m^{-1}$ for all $m \in \mathbb{Z}$.

$$\begin{aligned}\Phi^{-1}(d_0+m) - \Phi^{-1}(c_0+m) &= \int_{c_0+m}^{d_0+m} w(\tau)\, d\tau \\ &\leq w(m) \cdot \int_{c_0}^{d_0} v(\tau)\, d\tau = V(c_0,d_0)w(m).\end{aligned}$$

Therefore,

$$\Phi^{-1}(d_0+m) - \Phi^{-1}(c_0+m) \leq w(m)/\widetilde{a_w} \leq a_m^{-1},$$

as per the assumption. Since $\theta \in \mathbf{L}^2(\mathbb{R})$ with compact support implies $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$, we can apply Theorem 4.3(ii) to finish the proof. □

In fact, without additional assumptions on the warping function Φ, the condition $a_m \leq \widetilde{a_w}/w(m)$ in Corollary 4.4 cannot be improved. To see this we construct a warping function, such that any choice $a_m > \widetilde{a_w}/w(m)$ yields $a_m^{-1} < \Phi^{-1}(d) - \Phi^{-1}(c)$, for all nonempty, closed intervals $[c,d]$. Hence, the conditions of Theorem 4.3(ii) are violated. Choose $\Phi = \log$ and note $\Phi^{-1}(\tau) = e^\tau = w(\tau)$ for all $\tau \in \mathbb{R}$. We can choose $v = w$ and obtain

$$e^{d_0+m} - e^{c_0+m} = e^m \int_{c_0}^{d_0} e^\tau d\tau = V(c_0,d_0)w(m) = (\Phi^{-1}(d_0) - \Phi^{-1}(c_0))v(m),$$

to show that for a logarithmic warping function the natural decimation factors are indeed the coarsest possible decimation factors to satisfy the painless case conditions.

Our final set of sufficient Bessel and frame conditions is concerned with the cases that θ is not compactly supported, but still sufficiently localized, or that larger decimation factors, not permitted by Theorem 4.3(ii), are desired. In this setting, the verification of the frame property becomes substantially harder. To obtain a sufficient condition, it is possible to estimate the alias terms $\sum_{l\neq 0} |g_m \overline{\mathbf{T}_{la_m^{-1}} g_m}|$, $m \in \mathbb{Z}$, in the Walnut representation of the frame operator of $\mathcal{G}(\Phi,\theta,\mathbf{a})$, see [41, Proposition 3.7]. The goal is to provide a decay condition on θ and a density condition on the decimation factors $\mathbf{a}$, such that the conditions of [41, Proposition 3.7] are guaranteed to be satisfied. Note that these conditions were recently improved by Lemvig et al. [43], under the additional, separate assumption of the so-called α-local integrability condition. However, our results do not benefit from the sharper condition, such that we are content with relying on [41, Proposition 3.7].

One would be tempted to apply [19, Corollary 3.5], which uses decay of the g_m to determine a density condition on the a_m, to warped filter banks. That result imposes (at least) the following conditions on a filter bank $(g_{m,n})_{m,n\in\mathbb{Z}} \subset \mathbf{L}^{2,\mathcal{F}}(D)$ generated from $(g_m)_{m\in\mathbb{Z}} \subset \mathbf{L}^2(D)$ and $(a_m)_{m\in\mathbb{Z}} \subset \mathbb{R}^+$:

- There are constants $\tilde{A}, \tilde{B}$, such that $0 < \tilde{A} \leq \sum_{m\in\mathbb{Z}} |g_m(\xi)|^2 \leq \tilde{B} < \infty$, almost everywhere (on D).
- There are $C, \epsilon > 0$ and a δ-separated set $(b_m)_{m\in\mathbb{Z}}$, i.e. $\inf_{k,m\in\mathbb{Z}} |b_m - b_k| \geq \delta > 0$, such that $|g_m(\xi)| \leq C(1+|\xi - b_m|)^{-(2+\epsilon)}$ almost everywhere.

Note that the parameter set $(p_k)_{k\in\mathbb{Z}}$ and $(C_k)_{k\in\mathbb{Z}}$ in [19, Corollary 3.5] are required to be bounded above and below with $p_k > 2$ for all k. Hence, if the conditions above are violated, the conditions of [19, Corollary 3.5] are surely violated as well. However, these conditions pose severe restrictions for warped filter banks. In fact, under reasonable assumptions on $\mathcal{G}(\Phi,\theta,\mathbf{a})$, they imply that $w = (\Phi^{-1})'$ is bounded above and thus Φ must have at least linear asymptotic growth.

Proposition 4.5. *Let $\mathcal{G}(\Phi,\theta,\mathbf{a})$ be a warped filter bank for $\mathbf{L}^{2,\mathcal{F}}(D)$, with nonzero prototype θ, and set $b_m = \Phi^{-1}(m)$, for all $m \in \mathbb{Z}$. Then the following hold:*

(i) If the open interval D is a true subset of $\mathbb{R}$, i.e. $D \subsetneq \mathbb{R}$, then for any $\delta > 0$, there is an $m \in \mathbb{Z}$, such that $|b_{m+1} - b_m| \leq \delta$.

(ii) If there are $C_0 > 0$, $p_0 > 1/2$, such that $\theta \leq C_0(1 + |\cdot|)^{-p_0}$ and there are constants $\tilde{A}, \tilde{B}$, such that $0 < \tilde{A} \leq \sum_{m\in\mathbb{Z}} |g_m|^2 \leq \tilde{B} < \infty$ almost everywhere, then $\sup_{m\in\mathbb{Z}} a_m < \infty$ and $\limsup_{m\to\infty} a_m \neq 0 \neq \limsup_{m\to-\infty} a_m$.

(iii) If the assumptions of (ii) hold and additionally there are constants $p, C > 0$, such that $|g_m| \leq C(1 + |(\cdot) - b_m|)^{-p}$ almost everywhere, for all $m \in \mathbb{Z}$, then $w = (\Phi^{-1})' \in \mathbf{L}^\infty(\mathbb{R})$.

Proof. Ad (i): Assume without loss of generality that D is bounded below with $\inf\{\xi : \xi \in D\} = c \in \mathbb{R}$. Clearly, since Φ' is is continuous and positive, $\lim_{\xi\to c} \Phi'(\xi) = \infty$, implying $\lim_{\tau\to-\infty}(\Phi^{-1})'(\tau) = 0$ and (i) easily follows.

Ad (ii): Without loss of generality, assume $\operatorname{ess\,sup}_{\tau\in\mathbb{R}} \theta(\tau) = 1$. Then $\sup_{m\in\mathbb{Z}} a_m = \infty$ or $\lim_{m\to-\infty} a_m = \infty$ easily imply that a finite upper bound for $\sum_{m\in\mathbb{Z}} |g_m|^2 = \sum_{m\in\mathbb{Z}} a_m|\theta(\Phi(\cdot)-m)|^2$ cannot be found. Let $\overline{a} := \sup_{m\in\mathbb{Z}} a_m < \infty$. For every $\epsilon > 0$, there is $m_\epsilon \in \mathbb{Z}$, such that

$$\sum_{m\leq(k-m_\epsilon)} a_m|\theta(\tau - m)|^2 \leq C_0^2 \sum_{m\leq(k-m_\epsilon)} \overline{a}(1 + |\tau - m|)^{-2p_0} \leq \epsilon/2, \text{ for almost every } \tau \geq k,\ k \in \mathbb{Z}.$$

If $\limsup_{m\to\infty} a_m = 0$, then there is a $k_\epsilon \in \mathbb{Z}$, such that $\sup\limits_{m>(k_\epsilon-m_\epsilon)} a_m \leq \epsilon/(2B)$, where the constant B is defined as

$$B := \operatorname*{ess\,sup}_{\tau\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\theta(\tau - m)|^2 \leq \operatorname*{ess\,sup}_{\tau\in\mathbb{R}} C_0^2 \sum_{m\in\mathbb{Z}} (1 + |\tau - m|)^{-2p_0} < \infty.$$

Together, we obtain

$$\sum_{m\in\mathbb{Z}} a_m|\theta(\tau - m)|^2 \leq \epsilon, \text{ for almost every } \tau \geq k_\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the desired lower bound $\tilde{A}$ cannot exist. $\limsup_{m\to-\infty} a_m \neq 0$ is proven by the same steps.

Ad (iii): We show that under the assumptions of (iii), $w \notin \mathbf{L}^\infty(\mathbb{R})$ implies $\theta \equiv 0$, which contradicts the assumption that θ is nonzero. Begin by noting that $|g_m| \leq C(1 + |(\cdot) - \Phi^{-1}(m)|)^{-p}$ is equivalent to

$$\theta \leq \frac{C}{\sqrt{a_m}(1 + |\Phi^{-1}(\cdot + m) - \Phi^{-1}(m)|)^p}, \text{ almost everywhere.}$$

Moreover, for all $\tau_0 \in \mathbb{R}$,

$$\Phi^{-1}(\tau_0 + m) - \Phi^{-1}(m) = \int_0^{\tau_0} w(m + \tau)\, d\tau \geq \int_0^{\tau_0} \frac{w(m)}{v(-\tau)}\, d\tau,$$

where we used v-moderateness of w. If $\tau \geq \tau_1 > 0$, then with $C_{\tau_1} := \int_0^{\tau_1} \frac{1}{v(-\tau)} d\tau$, we obtain by positivity of v that

$$\theta(\tau) \leq \frac{C}{\sqrt{a_m}(1 + |\Phi^{-1}(\tau + m) - \Phi^{-1}(m)|)^p} \leq \frac{C}{\sqrt{a_m}(1 + C_{\tau_1} w(m)|)^p}, \text{ for almost all } \tau \geq \tau_1. \quad (24)$$

Now, if $w \notin \mathbf{L}^\infty(\mathbb{R})$, then either $\lim_{m\to\infty} w(m) = \infty$ or $\lim_{m\to-\infty} w(m) = \infty$, by continuity of w. For the right hand side of (24) to be bounded below, this implies either $\lim_{m\to\infty} a_m = 0$ or $\lim_{m\to-\infty} a_m = 0$, which is prohibited by (ii). Since $\tau_1 > 0$ was arbitrary, we obtain that necessarily $\theta(\tau) = 0$ for almost every $\tau > 0$. An analogous argument shows that $\theta(\tau) = 0$ for almost every $\tau < 0$. Therefore, $\theta \equiv 0$, completing the proof by contradiction. □

Considering Proposition 4.5, a quick glance at Examples 3.6–3.9 shows that the requirements of [19, Theorem 3.4] are highly undesirable for warped filter banks. Instead, we establish a decay condition on θ that is only a mild restriction compared to the standard condition $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$. This condition ensures the Bessel property and, when complemented by sufficiently small decimation factors, even the frame property. The given result is of central interest, as it shows that warped filter banks also admit the construction of frames for many prototype filters θ with full support.

Theorem 4.6. *Let $\Phi : D \to \mathbb{R}$ be a warping function with $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$, fix an arbitrary $\epsilon > 0$ and let w_1, w_2 denote the following weights*

$$w_1 = (1 + |\cdot|)^{1+\epsilon} \quad \text{and} \quad w_2 = (1 + |V(0, \cdot)|)^{1+\epsilon},$$

where V is as defined in (23)*. If*

$$\theta \in \mathbf{L}^\infty_{w_1}(\mathbb{R}) \cap \mathbf{L}^\infty_{w_2}(\mathbb{R}) \quad \text{and} \quad a_m \leq \tilde{a}/w(m), \quad \text{for all } m \in \mathbb{Z} \text{ and some } \tilde{a} > 0, \quad (25)$$

then $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a Bessel sequence. If additionally, there is a constant $A_1 > 0$ such that

$$0 < A_1 \leq \sum_{m\in\mathbb{Z}} |\mathbf{T}_m\theta|^2 \text{ almost everywhere,}$$

then there is a constant $\tilde{a}_0 > 0$ such that $\mathcal{G}(\Phi, \theta, \mathbf{a})$ is a frame, whenever $a_m \leq \tilde{a}_0/w(m)$, for all $m \in \mathbb{Z}$.

We want to highlight that, since necessarily $\theta \in \mathbf{L}^2_{\sqrt{v}}(\mathbb{R})$, the decay condition in (25) is only slightly stronger than requiring the appropriate integrability in addition to square-integrability. To see this, note that $\mathbf{L}^\infty_{w_2}(\mathbb{R}) \subset \mathbf{L}^2_{\sqrt{v}}(\mathbb{R}) \cap \mathbf{L}^1_v(\mathbb{R})$, but the change of variable $t = V(0, \tau)$, with $(V(0, \cdot))'(\tau) = v(\tau)$ by definition,

$$\int_{I_V} (1 + |t|)\, dt = \int_{\mathbb{R}} v(\tau)(1 + |V(0, \tau)|)\, d\tau,$$

where $I_V := \{t \in \mathbb{R} \colon \exists\ \tau \in \mathbb{R}, \text{ such that } V(0, \tau) = t\}$. Straightforward calculations using the v-moderateness of $w = (\Phi^{-1})'$ show that I_V is an interval and if $\inf(D) = -\infty$ ($\sup(D) = \infty$), then $\inf(I_V) = -\infty$ ($\sup(I_V) = \infty$). Hence, if I_V is unbounded, which is usually the case, then $\mathbf{L}^\infty_{(1+|V(0,\cdot)|)}(\mathbb{R}) \not\subset \mathbf{L}^1_v(\mathbb{R})$.

Before we proceed to prove Theorem 4.6, we require two auxiliary results.

Lemma 4.7. *Let $\Phi : D \to \mathbb{R}$ be a warping function such that w is v-moderate. There is a bijective, increasing function $A_v : \mathbb{R} \to \mathbb{R}$, such that $A_v(0) = 0$ and for all $c \in \mathbb{R}^+$, we have*

$$\left|\Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)\right| \geq cw(\tau_0) \quad \Longrightarrow |\tau_1 - \tau_0| \geq \begin{cases} |A_v^{-1}(c)| & \text{if } \tau_1 \geq \tau_0 \\ |A_v^{-1}(-c)| & \text{else} \end{cases}, \text{ for all } \tau_0, \tau_1 \in \mathbb{R}.$$

Proof. If $\tau_1 \geq \tau_0$, then the assumptions yield

$$cw(\tau_0) \leq \Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0) = \int_{\tau_0}^{\tau_1} w(\tau) d\tau \leq w(\tau_0)(\tau_1 - \tau_0) \sup_{\tau \in [0, \tau_1 - \tau_0]} v(\tau).$$

Analogous, we obtain for $\tau_1 < \tau_0$ that $cw(\tau_0) \leq w(\tau_0)(\tau_0 - \tau_1) \sup_{\tau \in [\tau_1 - \tau_0, 0]} v(\tau)$.

The function $A_v : \mathbb{R} \to \mathbb{R}$, $\tau \mapsto \tau \sup_{\tau_0 \in [\tau, 0] \cup [0, \tau]} v(\tau_0)$ is continuous and strictly increasing and thus invertible. Moreover, the above derivations show that, for all $\tau_0, \tau_1 \in \mathbb{R}$,

$$c \leq \operatorname{sgn}(\tau_1 - \tau_0) A_v(\tau_1 - \tau_0),$$

as desired. □

Lemma 4.7 allows us to derive the following result which will be crucial for proving Theorem 4.6.

Lemma 4.8. *For a given warped filter bank $\mathcal{G}(\Phi, \theta, \mathbf{a})$, with a sequence $\mathbf{a} = (a_m)_{m \in \mathbb{Z}}$ of decimation factors, define*

$$\mathbf{P}(\xi) := \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) := \sum_{m \in \mathbb{Z}} \left(|\theta(\Phi(\xi) - m)| \cdot \sum_{\substack{k \in \mathbb{Z} \setminus \{0\} \\ \xi + k a_m^{-1} \in D}} \left|\theta(\Phi(\xi + k a_m^{-1}) - m)\right| \right), \text{ for all } \xi \in D. \quad (26)$$

If $\theta \in \mathbf{L}^\infty_{w_1}(\mathbb{R}) \cap \mathbf{L}^\infty_{w_2}(\mathbb{R})$, with w_1, w_2 as in Theorem 4.6, and $a_m \leq \tilde{a}/w(m)$, for all $m \in \mathbb{Z}$ and some $\tilde{a} > 0$, then

$$\operatorname*{ess\,sup}_{\xi \in D} \mathbf{P}(\xi) < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{\xi \in D} \mathbf{P}(\xi) \overset{\tilde{a} \to 0}{\longrightarrow} 0.$$

Proof. In our estimation, we use *Hurwitz' zeta function* [50] , $\zeta(q, s) = \sum_{k \in \mathbb{N}_0} (q + k)^{-s}$, repeatedly. The function $\zeta(s, q)$ is finite for all $q > 0, s > 1$ and tends towards zero for s fixed and $q \to \infty$ or vice versa. It can be estimated by

$$\zeta(q, s) = q^{-s} + \sum_{k \in \mathbb{N}} (q + k)^{-s} < q^{-s} + \int_{\mathbb{R}^+} (q + t)^{-s} \, dt = q^{-s} + (s - 1)^{-1} q^{1-s} \quad (27)$$

and similarly $\zeta(q,s) > (s-1)^{-1}q^{1-s}$. Note that v-moderateness of w implies, for $t \in D$,

$$
\begin{aligned}
\left|t-\Phi^{-1}(m)\right| = \left|\Phi^{-1}(\Phi(t))-\Phi^{-1}(m)\right| &= \left|\int_m^{\Phi(t)} w(\tau)\ d\tau\right| \\
&= \left|\int_0^{\Phi(t)-m} w(\tau+m)\ d\tau\right| \\
&\leq w(m)\left|\int_0^{\Phi(t)-m} v(\tau)\ d\tau\right| = w(m)|V(0,\Phi(t)-m)|. \qquad (28)
\end{aligned}
$$

Moreover, with $C_1 := \|\theta\|_{\mathbf{L}^\infty_{w_1}(\mathbb{R})} < \infty$,

$$
\operatorname*{ess\ sup}_{\tau\in\mathbb{R}} \sum_{m\in\mathbb{Z}} |\mathbf{T}_m\theta(\tau)| \leq 2C_1 \sum_{m\in\mathbb{N}_0} \frac{1}{(1+m)^{1+\epsilon}} = 2C_1\zeta(1,1+\epsilon) =: \tilde{B} < 2C_1(1+\epsilon^{-1}) < \infty. \qquad (29)
$$

We proceed to estimate the inner sum in (26). To that end, define

$$
P_m := \sum_{\substack{k\in\mathbb{Z}\setminus\{0\} \\ \xi+ka_m^{-1}\in D}} \left|\theta(\Phi(\cdot+ka_m^{-1})-m)\right|, \text{ for all } m \in \mathbb{Z}.
$$

Now, assume that $\theta \in \mathbf{L}^\infty_{w_2}(\mathbb{R})$ with $C_2 := \|\theta\|_{\mathbf{L}^\infty_{w_2}(\mathbb{R})} > 0$. Insert into the definition of P_m to see that

$$
\begin{aligned}
P_m(\xi) &\leq C_2 \sum_{\substack{k\in\mathbb{Z}\setminus\{0\} \\ \xi+ka_m^{-1}\in D}} \left|(1+|V(0,\Phi(\xi+ka_m^{-1})-m))|)^{-1-\epsilon}\right| \\
&\leq C_2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(1+\left|\frac{\xi+ka_m^{-1}-\Phi^{-1}(m)}{w(m)}\right|\right)^{-1-\epsilon},
\end{aligned}
$$

for almost every $\xi \in D$. Here, we used (28) with $t = \xi + ka_m^{-1}$ to obtain the second inequality.

For any pair (ξ,m), there is a unique $k_{(\xi,m)} \in \mathbb{Z}$ such that $\xi+k_{(\xi,m)}a_m^{-1}-\Phi^{-1}(m) \in [-(2a_m)^{-1},(2a_m)^{-1})$. Let

$$
M_\xi := \{m \in \mathbb{Z} \colon k_{(\xi,m)} = 0\} \quad \text{and} \quad M_\xi^\dagger := \mathbb{Z} \setminus M_\xi.
$$

First assume that $m \in M_\xi$, i.e. $\xi - \Phi^{-1}(m) \in [-(2a_m)^{-1},(2a_m)^{-1})$. We can split the sum by the sign of k to obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+\left|\frac{\xi+ka_m^{-1}-\Phi^{-1}(m)}{w(m)}\right|\right)^{-1-\epsilon}\\
&\leq\sum_{k\in\mathbb{N}_0}\left(1+\left|\frac{(2a_m)^{-1}+ka_m^{-1}}{w(m)}\right|\right)^{-1-\epsilon}+\sum_{k\in\mathbb{N}_0}\left(1+\left|-\frac{(2a_m)^{-1}+ka_m^{-1}}{w(m)}\right|\right)^{-1-\epsilon}\\
&=2\sum_{k\in\mathbb{N}_0}\left(\frac{w(m)+(2a_m)^{-1}+ka_m^{-1}}{w(m)}\right)^{-1-\epsilon}=(*).
\end{aligned}$$

If $a_m\leq\tilde{a}/w(m)$, then $|(2a_m)^{-1}+ka_m^{-1}|\geq|(1/2+k)w(m)/\tilde{a}|$ and

$$(*)\leq 2\sum_{k\in\mathbb{N}_0}\left(\frac{\tilde{a}+1/2+k}{\tilde{a}}\right)^{-1-\epsilon}=2\tilde{a}^{1+\epsilon}\sum_{k\in\mathbb{N}_0}(\tilde{a}+1/2+k)^{-1-\epsilon}=2\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2,1+\epsilon).$$

Consequently, we obtain that

$$P_m(\xi)\leq 2C_2\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2,1+\epsilon),$$

for almost every $\xi\in D$. Now, if $m\in M_\xi^\dagger$, then a similar estimation yields

$$\begin{aligned}
P_m(\xi)&\leq\left|\theta(\Phi(\xi+k_{(\xi,m)}a_m^{-1})-m)\right|+C_2\sum_{k\in\mathbb{Z}\setminus\{0,k_{(\xi,m)}\}}\left(1+\left|\frac{\xi+ka_m^{-1}-\Phi^{-1}(m)}{w(m)}\right|\right)^{-1-\epsilon}\\
&\leq C_2+2C_2\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2,1+\epsilon),
\end{aligned}$$

almost everywhere. These estimates can now be inserted into the expression (26) for $\mathbf{P}$:

$$\begin{aligned}
\mathbf{P}(\xi)&=\sum_{m\in\mathbb{Z}}|\theta(\Phi(\xi)-m)|\cdot P_m(\xi)\\
&\leq C_2\sum_{m\in M_\xi^\dagger}|\theta(\Phi(\xi)-m)|+2C_2\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2,1+\epsilon)\cdot\sum_{m\in\mathbb{Z}}|\theta(\Phi(\xi)-m)|,
\end{aligned}\tag{30}$$

for almost every $\xi\in D$. Additionally, by (27),

$$\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2,1+\epsilon)<\left(1+\frac{\tilde{a}+1/2}{\epsilon}\right)\left(\frac{\tilde{a}}{\tilde{a}+1/2}\right)^{1+\epsilon}.$$

Applying (29) to estimate both terms, yields

$$\begin{aligned}
\operatorname*{ess\,sup}_{\xi\in D}\mathbf{P}(\xi)&\leq\tilde{B}C_2\cdot\left(1+2\tilde{a}^{1+\epsilon}\zeta(\tilde{a}+1/2,1+\epsilon)\right)\\
&<2C_1C_2(1+\epsilon^{-1})\left(1+2\left(1+\frac{\tilde{a}+1/2}{\epsilon}\right)\left(\frac{\tilde{a}}{\tilde{a}+1/2}\right)^{1+\epsilon}\right)<\infty.
\end{aligned}$$

For the second assertion, we have to show convergence of the essential supremum to 0 for $\tilde{a} \to 0$. By definition, $m \in M_\xi^\dagger$ implies $|\xi - \Phi^{-1}(m)| \geq (2a_m)^{-1} \geq w(m)(2\tilde{a})^{-1}$. Hence, we can apply Lemma 4.7, with $\tau_0 = m,\ \tau_1 = \Phi(\xi)$ and $c = (2\tilde{a})^{-1}$ to obtain

$$|\Phi(\xi) - m| \geq \begin{cases} |A_v^{-1}\left(1/(2\tilde{a})\right)| & \text{if } \Phi(\xi) - m \geq 0, \\ |A_v^{-1}\left(-1/(2\tilde{a})\right)| & \text{else.} \end{cases}$$

We can rewrite, with $m_- := \max M_\xi^\dagger \cap (-\infty, \Phi(\xi)), m_+ := \min M_\xi^\dagger \cap (\Phi(\xi), \infty)$,

$$\begin{aligned}
\sum_{m \in M_\xi^\dagger} |\theta(\Phi(\xi) - m)| &\leq \sum_{k \in \mathbb{N}_0} |\theta(\Phi(\xi) - m_- + k)| + \sum_{k \in \mathbb{N}_0} |\theta(\Phi(\xi) - m_+ - k)| \\
&\leq C_1 \left(\sum_{k \in \mathbb{N}_0} \frac{1}{\left(1 + |A_v^{-1}\left(1/(2\tilde{a})\right)| + k\right)^{1+\epsilon}} + \sum_{k \in \mathbb{N}_0} \frac{1}{\left(1 + |A_v^{-1}\left(-1/(2\tilde{a})\right)| + k\right)^{1+\epsilon}} \right) \\
&= C_1 \cdot \sum_{j=0}^{1} \zeta\left(1 + |A_v^{-1}\left((-1)^j/(2\tilde{a})\right)|, 1 + \epsilon\right).
\end{aligned}$$

All in all, we obtain for $\mathbf{P}$ the estimate

$$\begin{aligned}
&\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) \\
&\leq C_1 C_2 \cdot \sum_{j=0}^{1} \zeta\left(1 + |A_v^{-1}\left((-1)^j/(2\tilde{a})\right)|, 1 + \epsilon\right) + 2\tilde{B} C_2 \tilde{a}^{1+\epsilon} \zeta(\tilde{a} + 1/2, 1 + \epsilon) \\
&< C_1 C_2 \left(4(1 + \epsilon^{-1}) \left(1 + \frac{\tilde{a} + 1/2}{\epsilon}\right) \left(\frac{\tilde{a}}{\tilde{a} + 1/2}\right)^{1+\epsilon} + \sum_{j=0}^{1} \frac{1 + \epsilon^{-1}(1 + |A_v^{-1}\left((-1)^j/(2\tilde{a})\right)|)}{(1 + |A_v^{-1}\left((-1)^j/(2\tilde{a})\right)|)^{1+\epsilon}} \right),
\end{aligned} \tag{31}$$

almost everywhere. Since $A_v^{-1}(\tau) \overset{\tau \to \pm\infty}{\longrightarrow} \infty$, we see that

$$\operatorname*{ess\,sup}_{\xi \in D} \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) \overset{\tilde{a} \to 0}{\longrightarrow} 0,$$

as desired, finishing the proof. □

The last term on the right hand side of (31) depends heavily on the moderating weight v through the function A_v^{-1} and without further specifying v, a useful estimate for A_v^{-1} is out of reach. For $v(\tau) = e^\tau$, cf. Example 3.6, we have $A_v(\tau) = \tau \max\{1, e^\tau\}$. Thus, on $\mathbb{R}^+$, A_v^{-1} equals the product logarithm, such that the right hand side of (31) will decay very slowly for $\tilde{a} \to 0$. However, it decays quickly for increasing ϵ and for our experiments, performed with compactly supported or exponentially decaying θ, we have never observed any significant dependence of the choice of $\tilde{a}$ on v, even if v is chosen to be (asymptotically) optimal, compare the results presented in Section 5.1, in particular the frame bound ratios reported in Table 1. On the other hand, the estimate (31) may be rather coarse. For a smooth bell function θ, e.g. a Gaussian, even the base estimates $\theta \leq C_0(1 + |\cdot|)^{-1-\epsilon}$ and $\theta \leq C_1(1 + |V(0, \cdot)|)^{-1-\epsilon}$ do not allow the simultaneous choice of small constants C_0, C_1 and a large decay rate ϵ.

With Lemma 4.8 in place, proving Theorem 4.6 only requires a few simple steps.

Proof of Theorem 4.6. We will show that, with suitable choices of $\tilde{a}$, the conditions of Theorem 4.6 enable the application of [41, Proposition 3.7], which, adapted to our setting, states that

$$B := \operatorname*{ess\,sup}_{\xi\in D}\left[\sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\frac{1}{a_m}|g_m(\xi)\overline{g_m}(\xi - l/a_m)|\right] < \infty, \tag{32}$$

is sufficient for $\mathcal{G}(\Phi,\theta,\mathbf{a})$ to be a Bessel sequence with bound B. If additionally

$$A := \operatorname*{ess\,inf}_{\xi\in D}\left[\sum_{m\in\mathbb{Z}}\frac{1}{a_m}\left(|g_m(\xi)|^2 - \sum_{l\neq 0}|g_m(\xi)\overline{g_m}(\xi - l/a_m)|\right)\right] > 0, \tag{33}$$

then $\mathcal{G}(\Phi,\theta,\mathbf{a})$ constitutes a frame for $\mathbf{L}^{2,\mathcal{F}}(D)$ with lower frame bound A.

The main observation is the following: For any given warped filter bank $\mathcal{G}(\Phi,\theta,\mathbf{a})$, we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} a_m^{-1}|g_m(\xi)|^2 \pm \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}\setminus\{0\}} a_m^{-1}|g_m(\xi)\overline{g_m(\xi + ka_m^{-1})}| \\
&= \sum_{m\in\mathbb{Z}}|\theta(\Phi(\xi)-m)|^2 \pm \sum_{m\in\mathbb{Z}}\left(|\theta(\Phi(\xi)-m)|\cdot\sum_{\substack{k\in\mathbb{Z}\setminus\{0\}\\ \xi+ka_m^{-1}\in D}}|\theta(\Phi(\xi+ka_m^{-1})-m)|\right) \\
&= \sum_{m\in\mathbb{Z}}|\theta(\Phi(\xi)-m)|^2 \pm \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi), \text{ for almost every } \xi\in D.
\end{aligned}$$

By Lemma 4.8, $\operatorname{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) < \infty$. Moreover, since $\theta\in\mathbf{L}^{\infty}_{w_1}(\mathbb{R})$, we obtain the estimate

$$\operatorname*{ess\,sup}_{\tau\in\mathbb{R}}\sum_{m\in\mathbb{Z}}|\mathbf{T}_m\theta(\tau)| \leq \tilde{B} < \infty$$

as per (29). In total, with B as in (32),

$$B = \operatorname*{ess\,sup}_{\xi\in D}\left(\sum_{m\in\mathbb{Z}}|\theta(\Phi(\xi)-m)|^2 + \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi)\right) \leq \tilde{B}^2 + \operatorname*{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) < \infty,$$

and $\mathcal{G}(\Phi,\theta,\mathbf{a})$ is a Bessel sequence by [41, Proposition 3.7]. Similarly, with A as in (33),

$$A = \operatorname*{ess\,inf}_{\xi\in D}\left(\sum_{m\in\mathbb{Z}}|\theta(\Phi(\xi)-m)|^2 - \mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi)\right) \geq A_1 - \operatorname*{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi).$$

By Lemma 4.8, there is a constant $\tilde{a}_0 > 0$, such that $a_m \leq \tilde{a}_0/w(m)$, for all $m\in\mathbb{Z}$, implies

$$\operatorname*{ess\,sup}_{\xi\in D}\mathbf{P}_{\Phi,\theta,\mathbf{a}}(\xi) < A_1.$$

Thus, by [41, Proposition 3.7], we have that $\mathcal{G}(\Phi, \theta, \mathbf{a})$ constitutes a frame. □

Theorem 4.6 is extremely useful for proving (a) the *existence of a safe region* in which decimation factors can be chosen freely and (b) that compact support of the prototype θ is not a necessity for obtaining warped filter bank frames.

4.1. Warped filter bank frames with $\theta \in \mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$

When going through the results just presented, we regularly use the moderateness of $w = (\Phi^{-1})'$ to obtain estimates $w(\tau + m) \leq w(m)v(\tau)$. Clearly, we can exchange the roles of w and v, to obtain estimates in terms of $v(m)$ instead of $w(m)$. With this simple change, we can recover all the presented results in the setting where $\theta \in \mathbf{L}^2_{\sqrt{w}}(\mathbb{R})$ and natural decimation factors take the form $a_m = \tilde{a}/v(m)$, for some $\tilde{a} > 0$. Adapting the proofs amounts to simply exchanging the roles of w and v. Furthermore, $V(\tau_0, \tau_1)$ must be exchanged for $W(\tau_0, \tau_1) := \Phi^{-1}(\tau_1) - \Phi^{-1}(\tau_0)$, in the statements and proofs of Corollary 4.4, Lemma 4.8 and Theorem 4.6.

4.2. On tight warped filter bank frames

Using the framework of warped filter banks, tight frames are easily realized via Theorem 4.3(ii). This is one of the major assets of the presented construction. Tight frames are important for various reasons, the most important surely being that they provide a perfect reconstruction system in which the synthesis frame equals the analysis system up to a constant. Hence, there is no need for computing and/or storing a dual frame, which might be highly inefficient. Furthermore, the usage of tight frames guarantees that the synthesis shares the properties of the analysis, e.g. in terms of time-frequency localization.

Note that Theorem 4.3 implies that, if the decimation factors $\mathbf{a}$ are majorized by a set of natural decimation factors, then any tight warped filter bank frame $\mathcal{G}(\Phi, \theta, \mathbf{a})$ must necessarily satisfy, for some $C > 0$,

$$\sum_{m \in \mathbb{Z}} |\mathbf{T}_m \theta|^2 = C, \text{ a.e.} \tag{34}$$

Moreover, under the conditions of Theorem 4.3(ii), Equation (34) is even equivalent to $\mathcal{G}(\Phi, \theta, \mathbf{a})$ forming a tight frame. Therefore, θ that satisfy (34) are the optimal starting point when aiming to construct warped filter bank frames with small frame bound ratio, i.e. $B/A \approx 1$.

Although surely not the only methods for obtaining functions satisfying (34), we highlight here two classical methods that provide both compact support, which is required to apply Theorem 4.3(ii), and a prescribed smoothness: B-splines [17] and windows constructed as a superposition of truncated cosine waves of different frequency [49]. The second class contains classical window functions such as the Hann, Hamming and Blackman windows. We now recall a procedure to construct such functions that also satisfy (34). The method has previously been reported and proven as [57, Theorem 1]:

Let $K \in \mathbb{N}$ and $c_k \in \mathbb{R}$ for $k \in \{0, 1, \ldots, K\}$, and define

$$\vartheta(\tau) := \sum_{k=0}^{K} c_k \cos(2\pi k \tau) \mathbf{1}_{[-1/2, 1/2)}. \tag{35}$$

Then for any integer $R > 2K$

$$\sum_{m\in\mathbb{Z}} \left| \vartheta\left(\frac{\tau - m}{R}\right)\right|^2 = Rc_0^2 + \frac{R}{2}\sum_{k=1}^{K} c_k^2, \quad \forall \tau \in \mathbb{R}; \tag{36}$$

i.e. the sum of squares of a system of regular translates $(\mathbf{T}_m\theta)_{m\in\mathbb{Z}}$, with $\theta = \vartheta(\cdot/R)$, is constant.

The construction above can be combined with Theorem 4.3(ii) to easily construct tight frames by choosing the decimation factors a_m to satisfy

$$a_m^{-1} \geq \Phi^{-1}(m + R/2) - \Phi^{-1}(m - R/2). \tag{37}$$

In the following, we will demonstrate this for some of the examples given in Section 3.

For the purpose of all the following examples, we choose ϑ according to (35) with $K = 1$ and $c_0 = c_1 = 1/2$, i.e. we can choose $R \geq 3$. This function is often called the *Hann* or raised cosine window. The Hann window is among the most popular finitely supported Gabor windows or filters for time-frequency signal analysis.

Example 4.9 ($\Phi(\xi) = \operatorname{sgn}(\xi)\log(1 + |\xi|)$). For this choice of Φ, (37) takes the form

$$a_m^{-1} \geq \operatorname{sgn}(m + R/2)(e^{|m+R/2|} - 1) - \operatorname{sgn}(m - R/2)(e^{|m-R/2|} - 1),$$

or equivalently

$$a_m^{-1} \geq \begin{cases} (e^{|m|+R/2} - 1) - (e^{|m|-R/2} - 1) = e^{|m|}(e^{R/2} - e^{-R/2}) & \text{for } |m| \geq R/2, \\ (e^{m+R/2} - 1) + (e^{-m+R/2} - 1) = e^{R/2}(e^{|m|} + e^{-|m|}) - 2 & \text{else,} \end{cases}$$

where the latter case concerns the filters where $\operatorname{supp}(\mathbf{T}_m\theta)$ is not contained in either $[0, \infty)$ or $(-\infty, 0]$. We see that in both cases, a_m^{-1} is majorized by $e^{|m|}$, up to a constant depending solely on R. If we set $R = 3$, then a tight frame is obtained by choosing

$$a_m = \begin{cases} e^{-|m|}(e^{3/2} - e^{-3/2})^{-1} \geq \frac{1}{4.26e^{|m|}} & \text{for } |m| \geq 2, \\ (e^{3/2}(e^1 + e^{-1}) - 2)^{-1} > \frac{1}{11.84} & \text{for } |m| = 1, \\ (2e^{3/2} - 2)^{-1} > \frac{1}{6.97} & \text{for } m = 0. \end{cases}$$

On the other hand, Corollary 4.4 yields $\widetilde{a_w} = (2e^{3/2} - 2)^{-1}$ and $w(m) = v(m) = e^{|m|}$, i.e. $a_m = 2e^{-|m|}(e^{3/2} - 1)^{-1}$ for all $m \in \mathbb{Z}$, which is slightly more conservative.

Example 4.10 ($\Phi_\alpha(\xi) = \operatorname{sgn}(\xi)((1 + |\xi|)^{1-\alpha} - 1)$). Let $p := 1/(1 - \alpha) \in \mathbb{N}$. Then (37) can be rewritten as

$$a_m^{-1} \geq \begin{cases} (1 + |m| + R/2)^p - (1 + ||m| - R/2|)^p & \text{for } |m| \geq R/2, \\ (1 + R/2 + m)^p + (1 + R/2 - m)^p - 2 & \text{else.} \end{cases}$$

If $\alpha = 1/2$, i.e. $p = 2$, and $R = 3$, evaluation of the above conditions yields a tight warped filter bank with

$$a_m = \begin{cases} \frac{1}{6+6|m|} & \text{for } |m| \geq 2, \\ \frac{2}{25} & \text{for } |m| = 1, \\ \frac{2}{21} & \text{for } m = 0. \end{cases}$$

In this setting, Corollary 4.4 yields $\widetilde{a_w} = \frac{4}{21}$ and $w(m) = 2 + 2|m|$, see also Example 3.9.

Example 4.11 ($\Phi(\xi) = \tan(\xi)$). With w and v as in Proposition 3.11(v), Corollary 4.4 yields $\widetilde{a_w} = (R^3/6 + 2R - 4)^{-1}$ and, with $R = 3$, the following set of almost optimal natural decimation factors

$$a_m = \widetilde{a_w}/w(m) = \frac{2 + 2m^2}{13}.$$

5. Warped filter banks for digital signals

In the following, we consider sequences $x \in \ell^2(\mathbb{Z})$, interpreted as the samples of signals sampled at frequency ξ_s Hz. The discrete time Fourier transform (DTFT) and its inverse are denoted in the same fashion as the continuous Fourier transform before, i.e. $\hat{x}(\xi) := \mathcal{F}x(\xi) = \sum_{\mathbb{Z}} x(l)e^{-2\pi i l\xi/\xi_s}$. Note that $x \in \ell^2(\mathbb{Z})$ implies $\hat{x} \in \mathbf{L}^2(\mathbb{T})$, therefore the inverse DTFT maps $\mathbf{L}^2(\mathbb{T})$, with $\mathbb{T} = \mathbb{R}/\xi_s\mathbb{Z}$, onto $\ell^2(\mathbb{Z})$ by $\check{y}(l) = \mathcal{F}^{-1}y(l) = \xi_s^{-1}\int_{\mathbb{T}} y(\xi)e^{2\pi i l\xi/\xi_s}\, d\xi$. Discrete translation and modulation operators are given as usual. Finally, let $\underline{M} := \{0, \ldots, M-1\}$.

The material in this section summarizes work previously presented in [39], with the goal of highlighting the application of the presented methods in digital signal analysis. We take this chance to update and, in places, clarify these previous results, as well as bringing the notation more in line with our other work on warped filter banks.

A *(M channel, analysis) filter bank* $(g_{m,n})_{m\in\underline{M},n\in\mathbb{Z}}$, generated by $(g_m)_{m\in\underline{M}} \subset \mathbf{L}^2(\mathbb{T})$ and $(a_m)_{m\in\underline{M}} \subset \mathbb{N}$ is the set of finite energy sequences

$$g_{m,n} := \mathbf{T}_{na_m}\widetilde{g_m}, \text{ for all } m \in \underline{M}, n \in \mathbb{Z}. \tag{38}$$

Filter bank frames on $\ell^2(\mathbb{Z})$ are defined analogous to those on $\mathbf{L}^{2,\mathcal{F}}(D)$.

We construct warped filters $g_m \in \mathbf{L}^2(\mathbb{T})$ by restricting the warping function Φ to $I_{\xi_s,D} := D \cap (-\xi_s/2, \xi_s/2]$, i.e. we construct filter banks for sequences $x \in \ell^2(\mathbb{Z})$ with $\mathrm{supp}(\hat{x}) \subset I_{\xi_s,D}$. The interval $(-\xi_s/2, \xi_s/2]$ is interpreted as one period of the torus $\mathbb{T}$.

Although not strictly necessary, we will assume that $\theta \in \mathcal{C}$ with $\mathrm{supp}(\theta) \subseteq [c,d] \subset \Phi(I_{\xi_s,D})$. Let

$$M_{\max} = \max\{m \in \mathbb{Z} \colon \Phi^{-1}(m+d) \leq \sup(I_{\xi_s,D})\}$$
$$M_{\min} = \min\{m \in \mathbb{Z} \colon \Phi^{-1}(m+c) > \inf(I_{\xi_s,D})\},$$

and define the frequency responses

$$g_m(\xi) := \sqrt{a_m}(\mathbf{T}_m\theta) \circ \Phi(\xi), \quad \text{for all } \xi \in \mathbb{T},\ m \in \{M_{\min}, \ldots, M_{\max}\}, \tag{39}$$

where the constants $a_m \in \mathbb{N}$ are free parameters that are only applied once the decimation factors have been selected. The support restriction of θ ensures that $\mathrm{supp}(g_m) \subseteq I_{\xi_s,D}$. Clearly, if $I_{\xi_s,D}$ is a strict subset of $(-\xi_s/2, \xi_s/2]$, then either $M_{\min}$ or $M_{\max}$ is not finite. Hence, for the construction of a filter bank with a finite number of channels, we only consider the filters g_m for $m \in \{m_{\min}, \ldots, m_{\max}\}$, where $-\infty < m_{\min}, m_{\max} < \infty$ satisfy $m_{\min} \geq M_{\min}$ and $m_{\max} \leq M_{\max}$.

In order to cover the full frequency range $I_{\xi_s,D}$, we need to design additional band-pass filters. We distinguish two cases:

(i) If $I_{\xi_s,D} = (-\xi_s/2, \xi_s/2]$, then

$$g_{m_{\max}+1}(\xi) := \left(a_{m_{\max}+1} \sum_{m \in \mathbb{Z} \setminus [m_{\min}, m_{\max}]} |(\mathbf{T}_m \theta) \circ \Phi(\xi)|^2 \right)^{1/2}, \text{ for all } \xi \in \mathbb{T}. \tag{40}$$

(ii) If $I_{\xi_s,D} \subsetneq (-\xi_s/2, \xi_s/2]$, then

$$\begin{aligned} g_{m_{\min}-1}(\xi) &:= \left(a_{m_{\min}-1} \sum_{m < m_{\min}} |(\mathbf{T}_m \theta) \circ \Phi(\xi)|^2 \right)^{1/2} \quad \text{and} \\ g_{m_{\max}+1}(\xi) &:= \left(a_{m_{\max}+1} \sum_{m > m_{\max}} |(\mathbf{T}_m \theta) \circ \Phi(\xi)|^2 \right)^{1/2}, \text{ for all } \xi \in I_{\xi_s,D}, \end{aligned} \tag{41}$$

and 0 elsewhere. Once again $a_{m_{\min}-1}, a_{m_{\max}+1} \in \mathbb{N}$ can be selected freely.

The final filter bank contains $M := m_{\max} - m_{\min} + 1$ filters (or $M := m_{\max} - m_{\min} + 2$ if $I_{\xi_s,D} \subsetneq (-\xi_s/2, \xi_s/2]$) and after shifting the index set by $m_s := -m_{\min}$ (or $m_s := -m_{\min} - 1$), we obtain the M-channel, discrete warped filter bank $(g_m, a_m)_{m \in \underline{M}}$. Since the decimation factors $a_m \in \mathbb{N}$ only act as a normalization factor in the definition of the g_m, they can easily be chosen (and varied) a posteriori. For some exemplary frequency responses derived from the warping functions introduced in Examples 3.6–3.9, see Figure 2a. Note that Φ_{sqrt} corresponds to Example 3.9 with $\alpha = 1/2$. In Figure 2b, we show time-frequency plots of a test signal with respect to the same warping functions.

Necessary and sufficient frame conditions for discrete warped filter banks are analogous to the continuous case. In particular,

$$0 < A/\xi_s \leq \sum_{m \in \mathbb{Z}} |(\mathbf{T}_m \theta) \circ \Phi(\xi)|^2 = \sum_{m \in \underline{M}} a_m^{-1} |g_m(\xi)|^2 \leq B/\xi_s < \infty, \text{ for all } \xi \in I_{\xi_s,D}, \tag{42}$$

is a necessary condition for $(g_m, a_m)_{m \in \underline{M}}$ to constitute a frame with frame bounds A, B.

Recall that we assume $\text{supp}(\theta) \subseteq [c, d]$ and define $M_{\text{bp}} = \{M-1\}$, if $I_{\xi_s,D} = (-\xi_s/2, \xi_s/2]$ and $M_{\text{bp}} = \{0, M-1\}$ otherwise. For all $m \in \underline{M} \setminus M_{\text{bp}}$ choose the decimation factors a_m such that

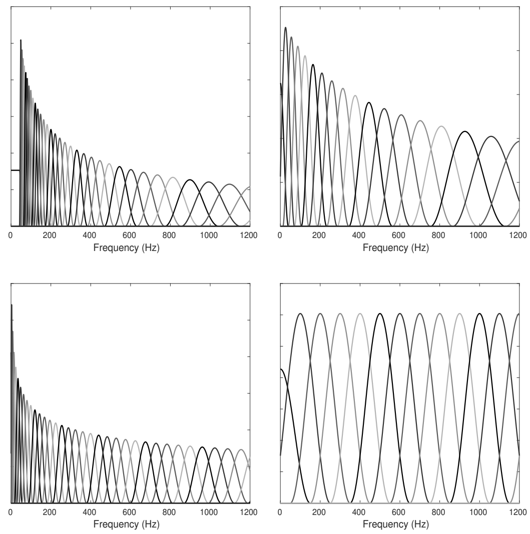
$$\xi_s/a_m \geq \Phi^{-1}(d + m + m_s) - \Phi^{-1}(c + m + m_s). \tag{43}$$

If additionally

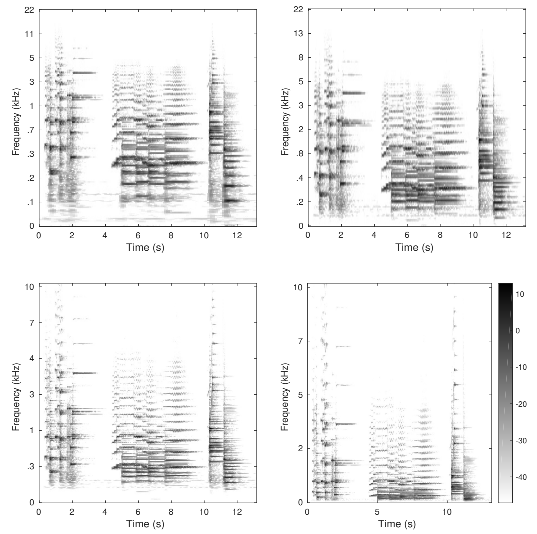
$$\begin{aligned} &\xi_s/a_{M-1} \geq \Phi^{-1}(d + m_s) + \xi_s - \Phi^{-1}(c + M + m_s - 1), \text{ if } I_{\xi_s,D} = (-\xi_s/2, \xi_s/2], \text{ or} \\ &\xi_s/a_0 \geq \Phi^{-1}(d + m_s) - \inf(I_{\xi_s,D}) \text{ and } \xi_s/a_{M-1} \geq \sup(I_{\xi_s,D}) - \Phi^{-1}(c + M + m_s - 1), \end{aligned} \tag{44}$$

then (42) is equivalent to the frame property (with bounds A, B) and a dual filter bank is obtained analogous to Theorem 4.3(ii).

The redundancy of a filter bank with full frequency range $(-\xi_s/2, \xi_s/2]$ is given by $\sum_{m \in \underline{M}} a_m^{-1}$. Since warped filter banks only cover $I_{\xi_s,D}$, the redundancy of $(g_{m,n})_{m \in \underline{M}, n \in \mathbb{Z}}$ is more accurately represented by $C_{\text{red}} = \frac{\xi_s}{\mu(I_{\xi_s,D})} \cdot \sum_{m \in \underline{M}} a_m^{-1}$. Selecting minimal decimation factors according to (43) and (44) results in $C_{\text{red}} \approx d - c$, i.e. the redundancy depends solely on the amount of overlap



(a) Frequency responses of warped filters (with low-pass filter (40)). The visualization was restricted to the frequency range 0 Hz–1.2 kHz.



(b) Time-frequency plots of a short piano and violin excerpt. Color indicates intensity is in dB, the colorbar is valid for all plots.

Figure 2: Warped filter bank examples: (top-left) $\Phi_{\log}(\xi) = 10\log(\xi)$, (top-right) $\Phi_{\text{erb}}(\xi) = 9.265\,\text{sgn}(\xi)\log(1 + |\xi|/228.8)$, (bottom-left) $\Phi_{\text{sqrt}}(\xi) = \text{sgn}(\xi)(\sqrt{1+|\xi|} - 1)$, (bottom-right) $\Phi_{\text{lin}}(\xi) = \xi/100$. Placement applies to subfigures (a) and (b). For subfigure (b), warping functions were scaled using (8) with $a = 4$, to increase the the filter density.

between the translates $\mathbf{T}_m\theta$.

If reduced redundancy is desired, we have obtained favorable results by reducing a_m, for $m \in \underline{M} \setminus M_{\text{bp}}$, by roughly a constant factor $\beta \in (0,1)$, i.e. choose a_m by rounding $\beta\xi_s(\Phi^{-1}(d+m+m_s) - \Phi^{-1}(c+m+m_s))^{-1}$ to the next integer, instead of using (43). This scheme is inspired by the results of the previous section that suggest to choose a_m that are majorized by (and rather close to) a set of natural decimation factors. The bandpass filters g_m, $m \in M_{\text{bp}}$ may have large plateaus and in that case, reducing the limits in (44) significantly usually leads to quickly deteriorating frame bounds. Therefore, a_m for $m \in M_{\text{bp}}$ have to be tuned more carefully.

To ensure the frame property for $(g_m, a_m)_{m\in\underline{M}}$ (and obtain suboptimal frame bounds), one can verify that

$$\infty > \sum_{m\in\underline{M}} a_m^{-1}|g_m(\xi)|^2 > \sum_{m\in\underline{M}} \left(a_m^{-1}|g_m(\xi)| \sum_{k=1}^{a_m-1} |\mathbf{T}_{k\xi_s/a_m}\overline{g_m}(\xi)| \right) =: \mathcal{A}(\xi) > 0, \text{ almost everywhere.} \tag{45}$$

The condition above is simply the application of [41, Proposition 3.7] to our setting. Since $\mathcal{A}$ depends continuously on θ, this shows that discrete warped filter bank frames can be obtained, even if the decimation regime given by (43) and (44) is not strictly satisfied.

Table 1: Frame bound ratios of warped filter banks from Figure 2a with varying redundancy. Columns correspond to warped filter banks with *approximately* equal redundancy. Numbers in parentheses are estimates obtained by considering the sum and difference, respectively, of the terms in (45).

$C_{\text{red}}(\approx)$	3	2	1.5	1.25	1.125
Φ_{lin}	1.000	1.220 (1.234)	1.961 (1.982)	3.880 (4.759)	6.868 (10.042)
Φ_{sqrt}	1.003	1.237 (1.243)	1.980 (1.997)	3.938 (4.894)	7.315 (11.135)
Φ_{erb}	1.000	1.240 (1.249)	1.970 (2.134)	3.860 (5.023)	7.122 (11.323)
Φ_{log}	1.014	1.240 (1.249)	1.973 (2.125)	3.876 (5.019)	7.159 (11.323)

5.1. Experiment: Frame bound ratio and estimates

The results in this section, as well as Figure 2, can be reproduced using the code provided at `http://ltfat.github.io/notes/049/`.

To illustrate that the redundancy can be significantly reduced we provide numerically computed frame bound ratios in Table 1, for different warped filter banks with varying redundancy. Additionally, estimates for the frame bound ratio obtained in the style of (45) are provided in parentheses. The filter banks were obtained from a Hann prototype, see Section 4.2, with $R = 3$ such that $\sum_m |\mathbf{T}_m\theta|^2 \equiv C$. The first column represents filter banks with decimation factors minimizing (43) and (44); for the remaining redundancies, decimation factors were chosen according to the reduced redundancy scheme with some $\beta < 1$. Even for redundancy as low as 9/8, the frame bound ratio[1] is significantly smaller than 10. Considering the estimate in the proof of Lemma 4.8, it is noteworthy that, for fixed redundancy, the dependence of the frame bound ratio on the warping function Φ seems to be quite limited.

Complementing the numerically obtained ratios in Table 1, we computed the estimates used to prove Theorem 4.6, with $\Phi_{\text{log}}(\xi) = 10\log(\xi)$ and $\theta = \vartheta(\cdot/3)$ with the Hann window ϑ as in (35) with $c_0 = c_1 = 1/2$. Hence, $\sum_m |\mathbf{T}_m\theta|^2 = 1.125$ by (36). Since θ is compactly supported, ϵ in the estimate (31) is arbitrary, but different choices lead to a different constant C_1C_2. We only considered the setting $a_m = \widetilde{a_w}/w(m)$ with $\widetilde{a_w}$ as in Corollary 4.4, where we know that in fact $P_{\Phi,\theta,\mathbf{a}} \equiv 0$, see (26) for the definition of $P_{\Phi,\theta,\mathbf{a}}$. For $\epsilon = 1, 2, 3, 4, 5$, Equation (31) yields the upper bounds 10.2, 6.5, 6.6, 8.2, 11.6 (rounded down to the first decimal) for $P_{\Phi,\theta,\mathbf{a}}$, such that Theorem 4.6 would not be sufficient to confirm the frame property, although the considered system is even tight. It might be interesting to note that, for small ϵ, the dominant quantity in (31) is the first term in parentheses, while for larger ϵ, the constant C_1C_2 is dominant. Interestingly, the term depending on A_v^{-1} had relatively minor contribution in our experiments.

6. Conclusion and Outlook

In this contribution, we have introduced a novel, flexible family of structured time-frequency filter banks. These warped filter banks are able to recreate or imitate important classical time-frequency representations, while providing additional design freedom. Warped filter banks allow

[1]In fact, due to a bug in older versions of the LTFAT Toolbox (`ltfat.github.io`) used for the frame bound calculations, the frame bound ratios reported in [39] are too large. In Table 1, corrected values are shown alongside ratios for lower redundancies not tested before.

for intuitive handling and the application of important results from the theory of generalized-shift invariant frames. In particular, the construction of tight frames of bandlimited filters reduces to the selection of a compactly supported prototype function whose integer translates satisfy a simple summation condition and sufficiently small decimation factors a_m. Moreover, the warping construction induces a natural choice of decimation factors that further simplifies the design of warped filter bank frames. With several examples, we have illustrated not only the flexibility of our method when selecting a non-linear frequency scale, but also the ease with which tight frames or snug frames can be constructed.

The complementary manuscript [40] discusses warped time-frequency representations in the context of continuous frames, determines the associated coorbit spaces and the warped time-frequency representations' sampling properties in the context of atomic decompositions and Banach frames [27, 28]. Future work will continue to explore practical applications of warped time-frequency representations and their finite dimensional equivalents on $\mathbb{C}^L$, as well as extending the warping method to multidimensional signals.

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